Solutions to Wiskunde voor bedrijfseconomen

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Preface

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If you find mistakes or possible improvements, please send an e-mail to Wiskunde1-TiSEM@tilburguniversity.edu.

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Solutions to Chapter 1

Exercise 1.1

a) To sketch a linear function, we need two points on this line. Consider the function $y(x) = 3x+2$. First, we find two points which satisfy the function, *e.g.*,

$$\text{If } x = 0, \quad y = 3\cdot 0 + 2 = 2.$$
$$\text{If } y = 0, \quad 0 = 3x+2 \Rightarrow x = -\frac{2}{3}.$$

Then, we plot the points $(0,2)$ and $(-\frac{2}{3},0)$ and draw a straight line passing through these two points on a coordinate plane. Similarly, we do the same for $z(x) = -5x+4$. The graphs of $y(x)$ and $z(x)$ are shown in Chapter 7.

b) The zero of $y(x)$ is the solution of

$$3x+2 = 0 \Leftrightarrow 3x = -2$$
$$\Leftrightarrow x = -\frac{2}{3}.$$

And, the zero of $z(x)$ is given by

$$-5x+4 = 0 \Leftrightarrow -5x = -4$$
$$\Leftrightarrow x = \frac{4}{5}.$$

c) We use part b) and obtain that $(-\frac{2}{3},0)$ is the intersection point with the x-axis for the graph of the function $y(x)$. Similarly, $(\frac{4}{5},0)$ is the intersection point with the x-axis for the graph of the function $z(x)$.

d) It holds that $y(0) = 2$. Hence, the intersection point with the y-axis for the graph of the function $y(x)$ is $(0,2)$. Similarly, $(0,4)$ is the intersection point with the y-axis for the graph of the function $z(x)$.

e) To determine the intersection point, we solve

$$y(x) = z(x) \Leftrightarrow 3x+2 = -5x+4$$
$$\Leftrightarrow 8x = 2$$
$$\Leftrightarrow x = \frac{1}{4}.$$

Since $y(\frac{1}{4}) = z(\frac{1}{4}) = 2\frac{3}{4}$, the point of intersection is $\left(\frac{1}{4}, 2\frac{3}{4}\right)$.

Exercise 1.2

$a = \dfrac{9-4}{3-2} = 5$, which gives $y(x) = 5x + b$. Since we know that $y(2) = 5 \cdot 2 + b = 4$, we obtain that $b = -6$. Hence, $a = 5$ and $b = -6$.

Exercise 1.3

a) Since $y(x) = x^2 + 7x + 6$, $D = 49 - 24 = 25 > 0$. According to the *abc-formula*, the two zeros are given by

$$x = \frac{-7 \pm \sqrt{25}}{2} = -1 \text{ and } -6.$$

b) Since $y(x) = 4x^2 + 2x + 1$, $D = 4 - 16 = -12 < 0$. There are no zeros in this case.

Exercise 1.4

a) To sketch a quadratic function, we need to determine the zeros and/or the locations of extremum. In this exercise, we solve for the zeros only. The zeros of $y(x) = x^2 + 4x + 3$ are given by

$$x = \frac{-4 \pm \sqrt{16-12}}{2} = -1 \text{ and } -3.$$

Using the sign survey of $y(x)$, we know which regions of x give positive/negative function values. This information determines whether the quadratic function is U-shaped or mountain-shaped. Finally, we draw a U-shaped or mountain-shaped smooth curve which passes through $(-1, 0)$ and $(-3, 0)$. Similarly, we do the same for $z(x) = -x^2 + 6$. The zeros of $z(x)$ are $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$. The graphs of $y(x)$ and $z(x)$ are shown in Chapter 7.

b) The points of intersection are the solutions of

$$\begin{aligned} y(x) = z(x) \ &\Leftrightarrow x^2 + 4x + 3 = -x^2 + 6 \\ &\Leftrightarrow 2x^2 + 4x - 3 = 0. \end{aligned}$$

According to the abc-formula, we have

$$x = \frac{-4 \pm \sqrt{40}}{4} = -\frac{4}{4} \pm \frac{\sqrt{4}\sqrt{10}}{4} = -1 \pm \tfrac{1}{2}\sqrt{10}.$$

Since $z(-1 - \frac{1}{2}\sqrt{10}) = 2\frac{1}{2} - \sqrt{10}$ and $z(-1 + \frac{1}{2}\sqrt{10}) = 2\frac{1}{2} + \sqrt{10}$, the points of intersection are

$$(x, y) = \left(-1 - \frac{1}{2}\sqrt{10}, 2\frac{1}{2} - \sqrt{10}\right) \text{ and } (x, y) = \left(-1 + \frac{1}{2}\sqrt{10}, 2\frac{1}{2} + \sqrt{10}\right).$$

Exercise 1.5

a) $h(x) = f(x) - g(x) = (2x + 4) - (2x^2 + 3x + 4) = -2x^2 - x$.

$$\begin{aligned} h(x) = 0 \ &\Leftrightarrow \ -2x^2 - x = 0 \\ &\Leftrightarrow x(-2x - 1) = 0 \\ &\Leftrightarrow x = 0 \text{ or } (-2x - 1) = 0 \\ &\Leftrightarrow x = 0 \text{ or } x = -\tfrac{1}{2}. \end{aligned}$$

From the sign survey of $h(x)$, it follows that $f(x) \geq g(x)$ if $-\frac{1}{2} \leq x \leq 0$.

b) The points of intersection are given by $x = -1 \pm \frac{1}{2}\sqrt{10}$. (See Exercise 1.4). From the sign survey of $f(x) - g(x)$, it follows that $f(x) \geq g(x)$ if $x \leq -1 - \frac{1}{2}\sqrt{10}$ or $x \geq -1 + \frac{1}{2}\sqrt{10}$.

c) $h(x) = f(x) - g(x) = x^2 - (5x - 4) = x^2 - 5x + 4$.

$$\begin{aligned} h(x) = 0 &\Leftrightarrow x^2 - 5x + 4 = 0 \\ &\Leftrightarrow x = 1 \text{ or } x = 4. \end{aligned}$$

From the sign survey of $h(x)$, it follows that $f(x) < g(x)$ if $1 < x < 4$.

Exercise 1.6

a) For a quadratic function to have two zeros, we need $D = b^2 - 4ac > 0$. Since $y(x) = x^2 + px + 3$, $D = p^2 - 12$. $D > 0$ implies $p > \sqrt{12}$ or $p < -\sqrt{12}$.

b) For a quadratic function to have two zeros, we need $D = b^2 - 4ac > 0$. Since $y(x) = p^2x^2 + 2px + 1$, $D = 4p^2 - 4p^2 = 0$. The discriminant is always equal to zero no matter what p is. Therefore, no such p exists.

Exercise 1.7

a)

$$\begin{aligned} x^3 - 2x^2 + x = 0 &\Leftrightarrow x(x^2 - 2x + 1) = 0 \\ &\Leftrightarrow x(x-1)^2 = 0 \\ &\Leftrightarrow x = 0 \text{ or } x = 1. \end{aligned}$$

b)

$$\begin{aligned} x^4 - x^2 + x(x^2 - 1) = 0 &\Leftrightarrow x^2(x^2 - 1) + x(x^2 - 1) = 0 \\ &\Leftrightarrow (x^2 + x)(x^2 - 1) = 0 \\ &\Leftrightarrow x(x+1)(x^2 - 1) = 0 \\ &\Leftrightarrow x = 0, x = 1 \text{ or } x = -1. \end{aligned}$$

c) Substitute $x^2 = p$. Then $3p^2 - 7p + 2 = 0$ gives $p = 2$ or $p = \frac{1}{3}$. Hence, $x^2 = \frac{1}{3}$ gives $x = -\sqrt{\frac{1}{3}}$ or $x = \sqrt{\frac{1}{3}}$, while $x^2 = 2$ results in $x = -\sqrt{2}$ and $x = \sqrt{2}$.

Exercise 1.8

a) $8 = 2^3$. Hence, $p = 3$.

b) $8^{\frac{4}{3}} = (2^3)^{\frac{4}{3}} = 2^4$. Hence, $p = 4$.

c) $\sqrt{32} = \sqrt{2^5} = (2^5)^{\frac{1}{2}} = 2^{\frac{5}{2}}$. Hence, $p = \frac{5}{2}$.

d) $64^{-\frac{1}{2}} = (2^6)^{-\frac{1}{2}} = 2^{-3}$. Hence, $p = -3$.

Exercise 1.9

a) $x^2x^5yy^2 = x^{2+5}y^{1+2} = x^7y^3$. Hence, $p = 7$, $q = 3$.

b) $\dfrac{xx^{\frac{1}{3}}y^2}{x^{-\frac{2}{3}}y^{-1}} = x^1x^{\frac{1}{3}}y^2x^{\frac{2}{3}}y^1 = x^{1+\frac{1}{3}+\frac{2}{3}}y^{2+1} = x^2y^3$. Hence, $p = 2$, $q = 3$.

c) $(x^{-1}y^4)^2 = (x^{-1})^2(y^4)^2 = x^{-2}y^8$. Hence, $p = -2$, $q = 8$.

d) $x^{\frac{10}{6}}\sqrt[3]{x} = x^{\frac{10}{6}}x^{\frac{1}{3}} = x^{\frac{12}{6}} = x^2y^0$. Hence, $p = 2$, $q = 0$.

Exercise 1.10

a) $\dfrac{x^{\frac{1}{8}}x^{\frac{3}{4}}}{x^{\frac{1}{2}}x^{\frac{7}{8}}} = \dfrac{x^{\frac{7}{8}}}{x^{1\frac{3}{8}}} = x^{-\frac{1}{2}} = 2$ and hence, $x = 2^{-2} = \frac{1}{4}$.

b) $\frac{x^2 x^{\frac{1}{2}}}{8x^{\frac{1}{3}}} = x^{\frac{2}{3}}$ gives $x^{2\frac{1}{6}} = 8x^{\frac{2}{3}}$. Hence, $x^{1\frac{1}{2}} = 8$, which implies $x = 8^{\frac{2}{3}} = 4$.

Exercise 1.11
$a^{-x} = a^{0-x} = \frac{a^0}{a^x} = \frac{1}{a^x}$, with the first equality sign following from property 2, and the third from property 5.

Exercise 1.12
a)
$$\begin{aligned} 2^x = 4^{4x+6} &\Leftrightarrow 2^x = (2^2)^{4x+6} \\ &\Leftrightarrow 2^x = 2^{8x+12} \\ &\Leftrightarrow x = 8x+12 \\ &\Leftrightarrow 7x = -12 \\ &\Leftrightarrow x = -\tfrac{12}{7}. \end{aligned}$$

b)
$$\begin{aligned} 27^{2x} = (\frac{1}{3})^{-x+2} &\Leftrightarrow (3^3)^{2x} = (3^{-1})^{2-x} \\ &\Leftrightarrow 3^{6x} = 3^{x-2} \\ &\Leftrightarrow 6x = x-2 \\ &\Leftrightarrow x = -\tfrac{2}{5}. \end{aligned}$$

c)
$$\begin{aligned} \left(\frac{1}{4}\right)^{x^2-1} = 1 &\Leftrightarrow \left(\frac{1}{4}\right)^{x^2-1} = \left(\frac{1}{4}\right)^0 \\ &\Leftrightarrow x^2 - 1 = 0 \\ &\Leftrightarrow x^2 = 1 \\ &\Leftrightarrow x = -1 \text{ or } x = 1. \end{aligned}$$

Exercise 1.13
The x-coordinate of the point of intersection is a solution of

$$\begin{aligned} y_1(x) = y_2(x) &\Leftrightarrow 3^{x+2} = 24 + 3^x \\ &\Leftrightarrow 3^{x+2} - 3^x = 24 \\ &\Leftrightarrow 3^x(3^2 - 1) = 24 \\ &\Leftrightarrow 8(3^x) = 24 \\ &\Leftrightarrow 3^x = 3 \\ &\Leftrightarrow x = 1. \end{aligned}$$

Hence, the intersection point is $(1, y_1(1)) = (1, 27)$.

Exercise 1.14
$\log(\frac{1}{x}) = \log 1 - \log x = 0 - \log x = -\log x$, where the fist equality sign follows from property 2, and the second from property 4.

Exercise 1.15
a)
$$\log x + 2\log y = \log x + \log y^2 = \log xy^2.$$

b)
$$\log x + \log \frac{1}{y} - \log z = \log x + \log \frac{1}{y} + \log z^{-1}$$
$$= \log x + \log \frac{1}{y} + \log \frac{1}{z}$$
$$= \log(x \cdot \frac{1}{y} \cdot \frac{1}{z}) = \log \frac{x}{yz}.$$

Exercise 1.16

a)
$$\ln(x+7) + \ln(x+3) = 0 \Leftrightarrow \ln((x+7)(x+3)) = 0$$
$$\Leftrightarrow \ln((x+7)(x+3)) = \ln 1$$
$$\Leftrightarrow (x+7)(x+3) = 1$$
$$\Leftrightarrow x^2 + 10x + 21 = 1$$
$$\Leftrightarrow x^2 + 10x + 20 = 0$$
$$\Leftrightarrow x = \frac{-10 \pm \sqrt{20}}{2} = -5 \pm \sqrt{5}.$$

Notice that the equation is only defined for $x + 3 > 0$. Hence, for $x > -3$. Thereby, $x = -5 - \sqrt{5}$ is not part of the domain, and the only valid solution is $x = -5 + \sqrt{5}$.

b) Substitute ${}^{3}\log x$ by p. Then we obtain $p^2 + 6 = 5p$. Hence,

$$p^2 + 6 = 5p \Leftrightarrow p^2 + -5p + 6 = 0$$
$$\Leftrightarrow p = \frac{5 \pm \sqrt{25-24}}{2}$$
$$\Leftrightarrow p = \tfrac{5}{2} \pm \tfrac{1}{2}$$
$$\Leftrightarrow p = 2 \text{ or } p = 3.$$

Hence, ${}^{3}\log x = 2$, which gives $x = 3^2 = 9$, or ${}^{3}\log x = 3$, which gives $x = 3^3 = 27$. Hence, $x = 9$ or $x = 27$.

Exercise 1.17

${}^{36}\log 81 = \dfrac{{}^{6}\log 81}{{}^{6}\log 36} = \dfrac{{}^{6}\log 81}{2} = \frac{1}{2}\,{}^{6}\log 81 = {}^{6}\log(81^{\frac{1}{2}}) = {}^{6}\log 9$. Hence, $x = 9$.

Exercise 1.18

The break-even point is the point where $R(x) = C(x)$.

$$R(x) = C(x) \Leftrightarrow px = c + vx$$
$$\Leftrightarrow px - vx = c$$
$$\Leftrightarrow (p - v)x = c$$
$$\Leftrightarrow x = \frac{c}{p - v}.$$

Exercise 1.19

a) $q_d(p) = ap + b$. We know that $a \cdot 220 + b = 180$ and $a \cdot 160 + b = 240$. Subtracting these two equations gives $a \cdot 60 = -60$, which gives $a = -1$. Plugging this into one of the two equations results in $b = 400$. Hence, $q_d(p) = -p + 400$, $(0 \le p \le 400)$.

b) Similar to part a), $q_s(p) = 2p - 200$, $(p \ge 100)$.

c) See Chapter 7 for the figure.
d) We solve

$$\begin{aligned} q_d(p) = q_s(p) &\Leftrightarrow -p + 400 = 2p - 200 \\ &\Leftrightarrow 3p = 600 \\ &\Leftrightarrow p = 200. \end{aligned}$$

Hence, $(q, p) = (200, 200)$.

Exercise 1.20
Consider the equation $\frac{x}{3} - \frac{a}{x} = 2$. If $a = 0$, it is a linear equation. Recall that a linear equation has precisely one solution as long as the slope is different from 0. Therefore, the equation has precisely one solution if $a = 0$. If $a \neq 0$ and $x \neq 0$, we have

$$\begin{aligned} \frac{x}{3} - \frac{a}{x} = 2 \ &\Leftrightarrow \frac{x^2}{3} - a = 2x \\ &\Leftrightarrow \frac{x^2}{3} - 2x - a = 0, \end{aligned}$$

which becomes a quadratic equation. A quadratic equation has precisely one solution if $D = 0$. In this case, we get

$$D = 4 + \frac{4}{3}a = 0 \ \Rightarrow \ a = -3.$$

As a result, the equation $\frac{x}{3} - \frac{a}{x} = 2$ has precisely one solution if $a = 0$ or $a = -3$.

Exercise 1.21
Since the demand function is $q = 60 - 10p$, the inverse demand function is $p = 6 - \frac{1}{10}q$, and total revenue is given by

$$TR(q) = pq = -\frac{1}{10}q^2 + 6q,$$

and total costs are

$$C(q) = 25 + 2q.$$

At the break-even point, we have

$$\begin{aligned} TR(q) = C(q) \ &\Leftrightarrow \ -\frac{1}{10}q^2 + 6q = 25 + 2q \\ &\Leftrightarrow \frac{1}{10}q^2 - 4q + 25 = 0 \\ &\Leftrightarrow q = \frac{4 \pm \sqrt{6}}{\frac{2}{10}} = 20 \pm 5\sqrt{6}. \end{aligned}$$

For both quantities we have a positive price, which implies that we have two break-even points. They are $q = 20 - 5\sqrt{6}$ and $q = 20 + 5\sqrt{6}$.

Exercise 1.22
$h(x) = x^3 + 2x - 3x^2 = x^3 - 3x^2 + 2x$.

$$\begin{aligned} h(x) = 0 &\Leftrightarrow x^3 - 3x^2 + 2x = 0 \\ &\Leftrightarrow x(x^2 - 3x + 2) = 0 \\ &\Leftrightarrow x = 0 \text{ or } x^2 - 3x + 2 = 0 \\ &\Leftrightarrow x = 0 \text{ or } x = \frac{3 \pm \sqrt{1}}{2} \\ &\Leftrightarrow x = 0 \text{ or } x = 1 \text{ or } x = 2. \end{aligned}$$

From the sign survey of $h(x)$, it follows that $x^3 + 2x \leq 3x^2$ if $x \leq 0$ or $1 \leq x \leq 2$.

Exercise 1.23
a) According to the abc-formula, the zeros of $y(x) = 2x^2 + 12x + 18$ are given by

$$x = \frac{-12 \pm \sqrt{144 - 4(2)(18)}}{4} = -3.$$

b) Since $y(x) = -x^2 - x + p = 0$, $D = 1 + 4p$. In order to have two zeros, D has to be greater than 0, which implies $p > -\frac{1}{4}$.
c) Two graphs intersect if there exists at least one solution of $y_1(x) = y_2(x)$.

$$\begin{aligned} y_1(x) = y_2(x) &\Leftrightarrow \frac{1}{4}x^2 - 5x + 6 = 3x + p \\ &\Leftrightarrow \frac{1}{4}x^2 - 8x + 6 - p = 0. \end{aligned}$$

To ensure no intersections, the discriminant of the above quadratic equation should be less than 0. As a result, we have

$$D = 64 - 6 + p < 0 \Rightarrow p < -58.$$

Exercise 1.24
a) The point of intersection is given by

$$\begin{aligned} y_1(x) = 3 &\Leftrightarrow {}^2\log(x - 2) = 3 \\ &\Leftrightarrow 2^{^2\log(x-2)} = 2^3 \\ &\Leftrightarrow x - 2 = 8 \\ &\Leftrightarrow x = 10. \end{aligned}$$

From the sign survey of $y_1(x) - 3$, it follows that $y_1(x) > 3$ if $x > 10$.

b) $h(x) = {}^2\log(x-2) + {}^2\log(x+4) - 2$.

$$\begin{aligned} h(x) = 0 &\Leftrightarrow {}^2\log(x-2) + {}^2\log(x+4) - 2 = 0 \\ &\Leftrightarrow {}^2\log(x-2) + {}^2\log(x+4) = 2 \\ &\Leftrightarrow {}^2\log((x-2)(x+4)) = 2 \\ &\Leftrightarrow {}^2\log(x^2+2x-8) = 2 \\ &\Leftrightarrow x^2+2x-8 = 4 \\ &\Leftrightarrow x^2+2x-12 = 0 \\ &\Leftrightarrow x = \frac{-2 \pm \sqrt{52}}{2} \\ &\Leftrightarrow x = -1 \pm \frac{\sqrt{52}}{2}. \\ &\Leftrightarrow x = -1 \pm \sqrt{13}. \end{aligned}$$

From the sign survey of $h(x)$, it follows that $y_1(x) < y_2(x)$ if $2 < x < -1 + \sqrt{13}$.

Exercise 1.25

a) $h(x) = \dfrac{2-x}{3+x} + x - 1$.

$$\begin{aligned} h(x) = 0 &\Leftrightarrow \frac{2-x}{3+x} = -x + 1 \\ &\Leftrightarrow 2 - x = (-x+1)(3+x) \\ &\Leftrightarrow 2 - x = -x^2 - 2x + 3 \\ &\Leftrightarrow x^2 + x - 1 = 0 \\ &\Leftrightarrow x = \frac{-1 \pm \sqrt{5}}{2}. \end{aligned}$$

From the sign survey of $h(x)$, it follows that $f(x) \geq g(x)$ if $-3 < x \leq -\frac{1}{2} - \frac{1}{2}\sqrt{5}$ or $x \geq -\frac{1}{2} + \frac{1}{2}\sqrt{5}$.

b) $h(x) = \dfrac{1-x^2}{3+x} - x - 1$.

$$\begin{aligned} h(x) = 0 &\Leftrightarrow \frac{1-x^2}{3+x} = x + 1 \\ &\Leftrightarrow 1 - x^2 = (x+1)(3+x) \\ &\Leftrightarrow 1 - x^2 = 4x + 3 + x^2 \\ &\Leftrightarrow 2x^2 + 4x + 2 = 0 \\ &\Leftrightarrow x^2 + 2x + 1 = 0 \\ &\Leftrightarrow (x+1)^2 = 0 \\ &\Leftrightarrow x = -1. \end{aligned}$$

From the sign survey of $h(x)$, it follows that $h(x)$ is negative if $-3 < x \leq -1$ or $x \geq -1$. Hence, we conclude that $f(x) \leq g(x)$ if $x > -3$.

Exercise 1.26

Note that x-axis can be represented by $y(x) = 0$. The x-coordinate of the intersection point of $y(x) = 0$ and $y(x) = e^x \ln(x + \frac{1}{2})$ is the solution of

$$\begin{aligned} e^x \ln(x + \frac{1}{2}) = 0 &\Leftrightarrow \underbrace{e^x}_{>0} \ln(x + \frac{1}{2}) = 0 \\ &\Leftrightarrow \ln(x + \frac{1}{2}) = 0 \\ &\Leftrightarrow e^{\ln(x+\frac{1}{2})} = e^0 \\ &\Leftrightarrow x + \frac{1}{2} = 1 \\ &\Leftrightarrow x = \frac{1}{2}. \end{aligned}$$

Hence, the intersection point is $(x, y) = (\frac{1}{2}, 0)$.

Exercise 1.27

a) The points of intersection are given by

$$\begin{aligned} y_1(x) = y_2(x) &\Leftrightarrow \sqrt{2x + 3} = x \\ &\Rightarrow 2x + 3 = x^2 \\ &\Leftrightarrow x^2 - 2x - 3 = 0 \\ &\Leftrightarrow x = \frac{2 \pm \sqrt{4 + 12}}{2} \\ &\Leftrightarrow x = -1 \text{ or } x = 3. \end{aligned}$$

Since we took a square on both sides, we have to check our results: $\sqrt{2 \cdot -1 + 3} = 1 \neq -1$ and $\sqrt{2 \cdot 3 + 3} = 3$. Hence, the only intersection point is $(x, y) = (3, 3)$.

b) From the sign survey of $y_1(x) - y_2(x)$, it follows that $y_1(x) < y_2(x)$ if $x > 3$.

Exercise 1.28

a)

$$\begin{aligned} 8^{2x} \cdot (\tfrac{1}{4})^{4x^2} = 16 \cdot (\tfrac{1}{2})^{6x^2} &\Leftrightarrow (2^3)^{2x} \cdot (2^{-2})^{4x^2} = 2^4 \cdot (2^{-1})^{6x^2} \\ &\Leftrightarrow 2^{6x} \cdot 2^{-8x^2} = 2^4 \cdot 2^{-6x^2} \\ &\Leftrightarrow 2^{-8x^2+6x} = 2^{-6x^2+4} \\ &\Leftrightarrow -8x^2 + 6x = -6x^2 + 4 \\ &\Leftrightarrow -2x^2 + 6x - 4 = 0 \\ &\Leftrightarrow x^2 - 3x + 2 = 0 \\ &\Leftrightarrow (x - 1)(x - 2) = 0 \\ &\Leftrightarrow x = 1 \text{ or } x = 2. \end{aligned}$$

b) $h(x) = (x^2 - 1)(x+2) + 3(x+1) \geq 0$

$$\begin{aligned} h(x) = 0 &\Leftrightarrow (x^2 - 1)(x+2) + 3(x+1) = 0 \\ &\Leftrightarrow (x+1)(x-1)(x+2) + 3(x+1) = 0 \\ &\Leftrightarrow (x+1)\Big((x-1)(x+2) + 3\Big) = 0 \\ &\Leftrightarrow (x+1)(x^2 + x + 1) = 0 \\ &\Leftrightarrow (x+1) = 0 \text{ or } (x^2 + x + 1) = 0. \end{aligned}$$

Then, $(x+1) = 0$ gives $x = -1$. Further, $x^2 + x + 1 = 0$ leads to $D = 1^2 - 4 \cdot 1 \cdot 1 = -3 < 0$. Hence, no zeros. Using a sign survey ($h(-2) = -3$ and $h(0) = 1$) we obtain that $x \geq -1$.

c) $h(x) = 2 \cdot {}^3\log(x) - {}^9\log(8x^2) + {}^9\log(81x) - 2 < 0$.

$$\begin{aligned} h(x) = 0 &\Leftrightarrow 2 \cdot {}^3\log(x) - {}^9\log(8x^2) + {}^9\log(81x) - 2 = 0 \\ &\Leftrightarrow {}^3\log(x^2) - {}^9\log(8x^2) + {}^9\log(81x) - {}^9\log(81) = 0 \\ &\Leftrightarrow {}^3\log(x^2) - {}^9\log(\frac{8x^2 \cdot 81}{81x}) = 0 \\ &\Leftrightarrow {}^3\log(x^2) - {}^9\log(8x) = 0 \\ &\Leftrightarrow \frac{{}^9\log(x^2)}{{}^9\log(3)} - {}^9\log(8x) = 0 \\ &\Leftrightarrow \frac{{}^9\log(x^2)}{\frac{1}{2}} - {}^9\log(8x) = 0 \\ &\Leftrightarrow 2 \cdot {}^9\log(x^2) - {}^9\log(8x) = 0 \\ &\Leftrightarrow {}^9\log(x^4) - {}^9\log(8x) = 0 \\ &\Leftrightarrow {}^9\log(\frac{x^4}{8x}) = 0 \\ &\Leftrightarrow {}^9\log(\frac{x^4}{8x}) = {}^9\log(1) \\ &\Leftrightarrow \frac{x^4}{8x} = 1 \\ &\Leftrightarrow x^3 = 8 \\ &\Leftrightarrow x = 2. \end{aligned}$$

By the sign survey ($h(1) = -\,{}^9\log 8 < 0$ and $h(3) = 2 - -\,{}^9\log 8 > 0$) we obtain that $0 < x < 2$.

2

Solutions to Chapter 2

Exercise 2.1

The difference quotient is given by

$$\begin{aligned}\frac{y(x+\Delta x)-y(x)}{\Delta x} &= \frac{y(1-3)-y(1)}{-3} \\ &= \frac{y(-2)-y(1)}{-3} \\ &= \frac{\Big((-2)^2+4(-2)+2\Big)-\Big(1^2+4\cdot 1+2\Big)}{-3} \\ &= \frac{-2-7}{-3} \\ &= 3.\end{aligned}$$

Exercise 2.2

Using $q(p) = -p^2 + 4p + 7$ and other information given in the question, we have

$$\begin{aligned}&\frac{q(3+\Delta p)-q(3)}{\Delta p} = \frac{1}{2} \\ \Leftrightarrow\ &\frac{(-(3+\Delta p)^2+4(3+\Delta p)+7)-(-3^2+4\cdot 3+7)}{\Delta p} = \frac{1}{2} \\ \Leftrightarrow\ &\frac{-(\Delta p)^2-6\Delta p-9+12+4\Delta p+7-10}{\Delta p} = \frac{1}{2} \\ \Leftrightarrow\ &\frac{-(\Delta p)^2-2\Delta p}{\Delta p} = \frac{1}{2} \\ \Leftrightarrow\ &-\Delta p-2 = \frac{1}{2} \\ \Leftrightarrow\ &\Delta p = -2\tfrac{1}{2}.\end{aligned}$$

Exercise 2.3

$$\begin{aligned}\frac{y(2+\Delta x)-y(2)}{\Delta x} &= \frac{\left((2+\Delta x)^2+4\right)-\left(2^2+4\right)}{\Delta x}\\ &= \frac{(\Delta x)^2+4\Delta x+4+4-8}{\Delta x}\\ &= \frac{(\Delta x)^2+4\Delta x}{\Delta x}\\ &= \Delta x+4.\end{aligned}$$

If $\Delta x \to 0$, then $\dfrac{y(2+\Delta x)-y(2)}{\Delta x} = 4$. Hence, $y'(2) = 4$.

Exercise 2.4
The tangent line is given by $y = ax + b$, with $a = y'(2) = 4$ (see Exercise 2.3). Hence, $y = 4x + b$. Furthermore, $8 = 4 \cdot 2 + b$, which implies that $b = 0$. Hence, the tangent line is given by $y = 4x$.

Exercise 2.5
a) $y(x) = 1, \quad y'(x) = 0.$
b) $y(x) = x^3, \quad y'(x) = 3x^2.$
c) $y(x) = \sqrt{x} = x^{\frac{1}{2}}, \quad y'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$
d) $y(x) = x^{-\frac{1}{3}}, \quad y'(x) = -\frac{1}{3}x^{-\frac{4}{3}}.$

Exercise 2.6
a) $D(p) = \frac{1}{p} = p^{-1}, \quad D'(p) = -p^{-2} = -\frac{1}{p^2}.$
Evaluated at $p = 3$, we have $D'(3) = -\frac{1}{3^2} = -\frac{1}{9}$.
b) $C(x) = xx^{\frac{1}{3}} = x^{\frac{4}{3}}, \quad C'(x) = \frac{4}{3}x^{\frac{1}{3}}.$
Evaluated at $x = 8$, we have $C'(8) = \frac{4}{3}(8)^{\frac{1}{3}} = 2\frac{2}{3}$.
c) $P(y) = 10^y, \quad P'(y) = 10^y \ln 10.$
Evaluated at $y = 0$, we have $P'(0) = \ln 10$.
d) $g(L) = {}^2\log L, \quad g'(L) = \frac{1}{L \ln 2}.$
Evaluated at $L = 4$, we have $g'(4) = \frac{1}{4 \ln 2}$.

Exercise 2.7
Since $y'(8) = -\frac{1}{48}$, the function value changes approximately by

$$\Delta y \approx y'(8) \cdot \Delta x = -\frac{1}{48} \cdot \frac{1}{2} = -\frac{1}{96}.$$

Exercise 2.8
The derivative of $y(x)$ is $y'(x) = \frac{3}{2}\sqrt{x}$. At $x = 4$, we have $y'(4) = 3$, and hence

$$\Delta x \approx \frac{\Delta y}{y'(4)} = \frac{\frac{1}{2}}{3} = \frac{1}{6}.$$

Exercise 2.9

a) Applying the sum rule, we have

$$\begin{aligned} y(x) &= 5x^3 - 15x^2 + 18x + 2, \\ y'(x) &= 15x^2 - 30x + 18. \end{aligned}$$

b) Applying the product rule twice, we have

$$\begin{aligned} y(x) &= (x-2)(x-7)(x-4), \\ y'(x) &= 1 \cdot (x-7)(x-4) + (x-2)((x-4)+(x-7)) \\ &= (x-7)(x-4) + (x-2)(x-4) + (x-2)(x-7). \end{aligned}$$

c) Applying the quotient rule, we have

$$\begin{aligned} y(x) &= \frac{(x+5)}{(x-3)}, \\ y'(x) &= \frac{(x-3)-(x+5)}{(x-3)^2} = -\frac{8}{(x-3)^2}. \end{aligned}$$

d) Rewriting $(e^x+3x)^2$ to $(e^x+3x)(e^x+3x)$ gives us the opportunity to use the product rule. Hence, we have

$$\begin{aligned} y(x) &= (e^x+3x)(e^x+3x), \\ y'(x) &= (e^x+3)(e^x+3x) + (e^x+3x)(e^x+3) = 2(e^x+3)(e^x+3x). \end{aligned}$$

Exercise 2.10

a) Using the quotient rule, we have

$$\begin{aligned} l(q) &= \frac{(2q^2+3)}{(3q+5)}, \\ l'(q) &= \frac{4q(3q+5)-3(2q^2+3)}{(3q+5)^2}, \\ l'(1) &= \frac{17}{64}. \end{aligned}$$

b) Using the quotient rule, we have

$$\begin{aligned} D(p) &= \frac{100}{p}, \\ D'(p) &= -\frac{100}{(p)^2}, \\ D'(5) &= -\frac{100}{25} = -4. \end{aligned}$$

c) Rewriting $(x^2+7)^3$ to $(x^2+7)(x^2+7)(x^2+7)$ gives us the opportunity to use the product rule twice. Hence, we have

$$\begin{aligned} C(x) &= (x^2+7)(x^2+7)(x^2+7), \\ C'(x) &= 3(x^2+7)^2(2x), \\ C'(1) &= 3(64)(2) = 384. \end{aligned}$$

Exercise 2.11

a) Using the sum rule and the product rule, we obtain

$$X(y) = y - y\ln y,$$
$$X'(y) = 1 - \ln y - \frac{y}{y} = -\ln y.$$

b) Using the product rule, we obtain

$$g(u) = u^2e^u,$$
$$g'(u) = 2ue^u + u^2e^u$$
$$= (u^2+2u)e^u.$$

c) Rewriting $L(1+\ln L)^2$ to $L(1+\ln L)(1+\ln L)$ gives us the opportunity to use the product rule twice. Hence, we have

$$P(L) = L(1+\ln L)(1+\ln L),$$
$$P'(L) = (1+\ln L)(1+\ln L) + L(\frac{1}{L}(1+\ln L) + (1+\ln L)\frac{1}{L})$$
$$= (1+\ln L)(3+\ln L).$$

Exercise 2.12

$MR(q) = R'(q) = 5\ln q + 5q\cdot\frac{1}{q} = 5\ln q + 5$. Hence, $MR(7) = 5\ln(7)+5$.

Exercise 2.13

a) $$\begin{cases} D(0)=10 \\ D(1)=8 \end{cases} \Leftrightarrow \begin{cases} b=10 \\ b-a=8 \end{cases} \Leftrightarrow \begin{cases} b=10 \\ a=2. \end{cases}$$

b) $$\begin{cases} D(0)=10 \\ D(10)=8 \end{cases} \Leftrightarrow \begin{cases} b=10 \\ b-10a=8 \end{cases} \Leftrightarrow \begin{cases} b=10 \\ a=0.2. \end{cases}$$

c) Since $MD(q) = D'(q) = -a$, the marginal demand is -2 in part a) and -0.2 in part b).

Exercise 2.14

a) $$\frac{D(2.525)-D(2.5)}{D(2.5)} = \frac{135\frac{125}{128} - 140.625}{140.625} = -\frac{119}{3600}.$$

Hence, $\%\Delta D = -3\frac{11}{36}$.

b) Since $D'(p) = -500+125p$, the elasticity of demand at $p = 2.5$ is

$$\epsilon = D'(2.5)\cdot\frac{2.5}{D(2.5)} = -3\tfrac{1}{3}.$$

c) $$\%\Delta D \approx \epsilon\cdot\%\Delta p \;\Leftrightarrow\; \%\Delta D \approx -3\tfrac{1}{3}\cdot 1\% = -3\frac{1}{3}.$$

d) $$\%\Delta D \approx \epsilon\cdot\%\Delta p \;\Leftrightarrow\; -1\% \approx -3\tfrac{1}{3}\%\Delta p \;\Leftrightarrow\; \%\Delta p \approx \tfrac{3}{10}.$$

Exercise 2.15

Since $D'(p) = -a\alpha p^{-\alpha-1}$, the price elasticity of demand is

$$\epsilon = D'(p)\frac{p}{D(p)} = -a\alpha p^{-\alpha-1}\frac{p}{ap^{-\alpha}} = -\alpha.$$

Exercise 2.16
$L(t) = 3t^{\frac{3}{2}}$, hence $L'(t) = 4\frac{1}{2}\sqrt{t}$, which gives $L'(3) = 4\frac{1}{2}\sqrt{3}$.

Exercise 2.17

a) $v(x) = 2x - 1$ and $u(v) = e^v$.
b) $v(x) = x^2 - 3$ and $u(v) = v^{-1}$.
c) $v(x) = x^2 + 4$ and $u(v) = \ln(v)$.

Exercise 2.18

a) $y(x) = e^{2x-1}$,
$y'(x) = e^{2x-1}2 = 2e^{2x-1}$.

b) $y(x) = (x^2 - 3)^{-1}$,
$y'(x) = (x^2 - 3)^{-2} \cdot (-2x) = -2x(x^2 - 3)^{-2}$.

c) $y(x) = \ln(x^2 + 4)$
$$y'(x) = \frac{1}{x^2 + 4} \cdot 2x = \frac{2x}{x^2 + 4}.$$

d) $y(x) = x^2 \ln(x^2 + 4)$.
$$y'(x) = 2x \ln(x^2 + 4) + x^2 \cdot \frac{1}{x^2 + 4} \cdot 2x = 2x \ln(x^2 + 4) + \frac{2x^3}{x^2 + 4}.$$
Note that here we have applied both the chain rule and the product rule.

Exercise 2.19

a) $r(q) = 2q\sqrt{2q - 1}$,
$$r'(q) = 2\sqrt{2q - 1} + \frac{4q}{2\sqrt{2q - 1}} = 2\sqrt{2q - 1} + \frac{2q}{\sqrt{2q - 1}}.$$
Evaluated at $q = 5$, we obtain
$$r'(5) = 2\sqrt{10 - 1} + \frac{10}{\sqrt{10 - 1}} = 6 + \frac{10}{3} = 9\tfrac{1}{3}.$$

b) $y(x) = e^{2x}$,
$y'(x) = 2e^{2x}$.
Evaluated at $x = 2$, we obtain
$y'(4) = 2e^4$

Exercise 2.20

a) To determine the inverse function $C(F)$, we solve $F(C) = \frac{9}{5}C + 32$ for C. We get
$$F = \frac{9}{5}C + 32 \Leftrightarrow C(F) = \frac{5}{9}F - 17\frac{7}{9}$$

b) Simply plugging in the functions gives
$$F(C(F)) = \frac{9}{5}(\frac{5}{9}F - 17\frac{7}{9}) + 32 = F - \frac{160}{5} + 32 = F$$
and
$$C(F(C)) = \frac{5}{9}(\frac{9}{5}C + 32) - 17\frac{7}{9} = C + \frac{160}{9} - \frac{160}{9} = C.$$

Exercise 2.21

a) $y(x) = x^3 \Leftrightarrow\ x(y) = y^{\frac{1}{3}}$.
b) $y(x) = \sqrt{x} \Leftrightarrow\ x(y) = y^2,\ y \geq 0$.
c) $y(x) = 2e^x \Leftrightarrow e^x = \frac{y}{2} \Leftrightarrow x(y) = \ln(\frac{y}{2})$.
d) $y(x) = x^2 + 4x + 4 = (x+2)^2, (x \geq 0) \Leftrightarrow x(y) = -2 + \sqrt{y}, (y \geq 4)$.

Exercise 2.22

a) We know that $y(1) = 2$. Hence, by the definition of the inverse, we also have $x(2) = 1$. Furthermore,

$$y'(x) = 3x^2 + 1 \qquad \Rightarrow y'(1) = 4$$

Then it follows that

$$x'(2) = \frac{1}{y'(1)} = \frac{1}{4}$$

b) We proceed as above. $y(1) = 2$ and thus $x(2) = 1$. The derivative is

$$y'(x) = 2 + \frac{1}{x} \qquad \Rightarrow y'(1) = 3$$

and thus

$$x'(2) = \frac{1}{y'(1)} = \frac{1}{3}$$

Exercise 2.23

a) The inverse demand function is $p(X) = 10 - 2X$.
b) The elasticity of demand is given by

$$\epsilon(X) = X'(p)\frac{p}{X(p)} = -\frac{1}{2} \cdot \frac{p}{5 - \frac{p}{2}} = \frac{-p}{10 - p} = \frac{p}{p - 10}.$$

c) The elasticity of inverse demand is

$$\epsilon(p) = p'(X)\frac{X}{p(X)} = -2 \cdot \frac{X}{10 - 2X} = \frac{-X}{5 - X} = \frac{X}{X - 5}.$$

d) At $p = 6$, The elasticity of demand is $\epsilon(X) = -\frac{2}{3}$. Since $X(6) = 2$, the elasticity of inverse demand at $X = 2$ is $\epsilon(p) = -\frac{3}{2}$. Hence, $\epsilon(X) = \frac{1}{\epsilon(p)}$ holds for $p = 6$ (and thus $X = X(6) = 2$).

Exercise 2.24

If the price increases from 4 to 9, the average increase in demand is

$$\frac{q(9) - q(4)}{9 - 4} = \frac{9\sqrt{9} - 4\sqrt{4}}{5} = \frac{19}{5} = 3\tfrac{4}{5}.$$

Exercise 2.25

Recall that difference quotient in x at a change Δx is defined by $\frac{y(x+\Delta x)-y(x)}{\Delta x}$. In this question, we have given $y(x) = x^2 + 5x + 3$ and the difference quotient is equal to 3 in $x = a$ at a change of $\Delta x = 3$. Using this information and the definition of difference quotient, we have

$$\begin{aligned}\frac{y(a+3)-y(a)}{3} = 3 &\Leftrightarrow \frac{((a+3)^2+5(a+3)+3)-(a^2+5a+3)}{3} = 3\\ &\Leftrightarrow a^2+6a+9+5a+15+3-a^2-5a-3 = 9\\ &\Leftrightarrow 6a+24 = 9\\ &\Leftrightarrow a = -\frac{15}{6} = -2\tfrac{1}{2}.\end{aligned}$$

Exercise 2.26

Since $y(x) = x^2 + 5x + 6$, the slope of the line is equal to

$$\frac{y(3)-y(1)}{3-1} = \frac{9+15+6-1-5-6}{2} = 9.$$

Exercise 2.27

Consider the function $y(x) = x^2 + 7$. With the help of the points $(a, y(a))$ and $(b, y(b))$, the slope of the line is equal to

$$\frac{y(b)-y(a)}{b-a} = \frac{b^2+7-(a^2+7)}{b-a} = \frac{b^2-a^2}{b-a} = \frac{(b-a)(b+a)}{b-a} = b+a.$$

From the exercise, we know that the slope is equal to 5 and $b - a = 3$. Hence, we have $b + a = 5$ and $b - a = 3$. Solving this system of equations gives

$$\begin{aligned}&\begin{cases} b+a=5\\ b-a=3\end{cases} \Leftrightarrow \begin{cases} b+a=5\\ b=3+a\end{cases} \Leftrightarrow \begin{cases} 3+a+a=5\\ b=3+a\end{cases} \Leftrightarrow \begin{cases} 2a=2\\ b=3+a\end{cases}\\ &\Leftrightarrow \begin{cases} a=1\\ b=3+a\end{cases} \Leftrightarrow \begin{cases} a=1\\ b=3+1\end{cases} \Leftrightarrow \begin{cases} a=1\\ b=4.\end{cases}\end{aligned}$$

Exercise 2.28

a) Consider the function $y(x) = \frac{x^2}{2x+1}$. The difference quotient of $y(x)$ in $x = 2$ at a change of $\Delta x = 3$ is

$$\frac{y(2+3)-y(2)}{3} = \frac{5^2/(10+1)-2^2/(4+1)}{3} = \frac{25/11-4/5}{3} = \frac{27}{55}.$$

b) Consider the function $y(x) = \frac{x^2}{2x+1}$. According to the definition of the difference quotient, the difference quotient of $y(x)$ in $x = 2$ at a change of Δx is given by

$$\begin{aligned}
\frac{y(2+\Delta x)-y(2)}{\Delta x} &= \left(\frac{(2+\Delta x)^2}{2(2+\Delta x)+1} - \frac{2^2}{2(2)+1}\right)/\Delta x \\
&= \left(\frac{4+4\Delta x+(\Delta x)^2}{5+2\Delta x} - \frac{4}{5}\right)\frac{1}{\Delta x} \\
&= \frac{(4+4\Delta x+(\Delta x)^2)5-4(5+2\Delta x)}{5(5+2\Delta x)}\frac{1}{\Delta x} \\
&= \frac{20+20\Delta x+5(\Delta x)^2-20-8\Delta x}{5(5+2\Delta x)\Delta x} \\
&= \frac{(5\Delta x+12)\Delta x}{5(5+2\Delta x)\Delta x} \\
&= \frac{5\Delta x+12}{5(5+2\Delta x)}.
\end{aligned}$$

As $\Delta x \to 0$, the difference quotient becomes

$$\frac{5\cdot 0+12}{5(5+2\cdot 0)} = \frac{12}{25}.$$

Exercise 2.29

a) Since $y(x) = x^{\frac{1}{2}}x^{\frac{1}{3}} = x^{\frac{5}{6}}$, the derivative of $y(x)$ is equal to

$$y'(x) = \frac{5}{6}x^{-\frac{1}{6}}.$$

Let the tangent line be $y = ax + b$. To determine a and b, we make use of the following system of equations:

$$\begin{cases} y'(1) = a \\ y(1) = a(1) + b \end{cases} \Leftrightarrow \begin{cases} \frac{5}{6}(1)^{-\frac{1}{6}} = a \\ 1^{\frac{5}{6}} = a + b \end{cases} \Leftrightarrow \begin{cases} a = \frac{5}{6} \\ 1 = a + b \end{cases} \Leftrightarrow \begin{cases} a = \frac{5}{6} \\ b = 1 - \frac{5}{6} = \frac{1}{6}. \end{cases}$$

As a result, the tangent line is given by

$$y = \tfrac{5}{6}x + \tfrac{1}{6}.$$

b) Since $y(x) = 2^{-x}$, the derivative of $y(x)$ is

$$y'(x) = -2^{-x}\ln 2.$$

Let the tangent line be $y = ax + b$. To determine a and b, we make use of the following system of equations:

$$\begin{cases} y'(1) = a \\ y(1) = a(1) + b \end{cases} \Leftrightarrow \begin{cases} -2^{-1}\ln 2 = a \\ 2^{-1} = a + b \end{cases} \Leftrightarrow \begin{cases} a = -\frac{1}{2}\ln 2 \\ b = \frac{1}{2} + \frac{1}{2}\ln 2. \end{cases}$$

As a result, the tangent line is equal to

$$y = -\frac{1}{2}(\ln 2)x + \frac{1}{2} + \frac{1}{2}\ln 2.$$

Exercise 2.30

Since $y(x) = x^2 + 3x + 4$, the derivative of $y(x)$ is

$$y'(x) = 2x + 3.$$

The slope of the line through the points $(0, 4)$ and $(2, 14)$ is given by

$$\frac{14-4}{2-0} = 5.$$

And, the slope of the tangent line at the tangent point $(x_0, y(x_0))$ is

$$a = y'(x_0) = 2x_0 + 3.$$

Because these slopes are the same, we have

$$5 = 2x_0 + 3 \iff x_0 = 1.$$

Exercise 2.31

Since $y(x) = 2x^2 + 2$, the derivative of $y(x)$ is

$$y'(x) = 4x.$$

Let the tangent line be $y = ax + b$. First, we know that the tangent line intersects the x-axis in $x = 1$. In other words, the point $(1, 0)$ satisfies $y = ax + b$:

$$0 = a(1) + b \iff a = -b$$

Hence, the tangent line becomes $y = ax - a$. To determine a, we make use of the following system of equations:

$$\begin{cases} y'(x_0) = a \\ y(x_0) = ax_0 - a \end{cases} \iff \begin{cases} 4x_0 = a \\ 2x_0^2 + 2 = ax_0 - a \end{cases}$$

$$\iff \begin{cases} a = 4x_0 \\ 2x_0^2 + 2 = (4x_0)x_0 - (4x_0) \end{cases}$$

$$\iff \begin{cases} a = 4x_0 \\ 2x_0^2 - 4x_0 - 2 = 0 \end{cases}$$

$$\iff \begin{cases} a = 4x_0 \\ x_0 = (4 \pm \sqrt{16+16})/4 \end{cases}$$

$$\iff \begin{cases} a = 4(1 \pm \sqrt{2}) \\ x_0 = 1 \pm \sqrt{2}. \end{cases}$$

As a result, the tangent lines are given by

$$y = 4(1+\sqrt{2})x - 4(1+\sqrt{2})$$

and

$$y = 4(1-\sqrt{2})x - 4(1-\sqrt{2}).$$

Exercise 2.32

Since $y(x) = x^3 + 3x^2 - 18x + 12$, the derivative of $y(x)$ is

$$y'(x) = 3x^2 + 6x - 18.$$

To find the point of tangency $(x_0, y(x_0))$, we make use of the following system of equations:

$$\begin{cases} y'(x_0) = 6 \\ y(x_0) = 92 + 6x_0 \end{cases} \Leftrightarrow \begin{cases} 3x_0^2 + 6x_0 - 18 = 6 \\ x_0^3 + 3x_0^2 - 18x_0 + 12 = 92 + 6x_0 \end{cases}$$
$$\Leftrightarrow \begin{cases} 3x_0^2 + 6x_0 - 24 = 0 \\ x_0^3 + 3x_0^2 - 24x_0 - 80 = 0. \end{cases}$$

Solving the first equation gives

$$3x_0^2 + 6x_0 - 24 = 0 \Leftrightarrow x_0 = \frac{-6 \pm \sqrt{36 + 288}}{6}$$
$$\Leftrightarrow x_0 = -4 \text{ or } x_0 = 2.$$

Next, we check whether these x_0's satisfy the second equation:

For $x_0 = -4$, $(-4)^3 + 3(-4)^2 - 24(-4) - 80 = 0 = 0.$
For $x_0 = 2$, $(2)^3 + 3(2)^2 - 24(2) - 80 = -104 \neq 0.$

Therefore, the only valid point of tangency is $(x, y) = (-4, y(-4)) = (-4, 68)$.

Exercise 2.33

Since $y(x) = x^3 + px^2 + 3x + 2 + p$, the derivative of $y(x)$ is

$$y'(x) = 2x^2 + 2px + 3.$$

The tangent line is horizontal when its slope is zero. Hence, the points of tangency are the solutions of $y'(x) = 0$:

$$y'(x) = 3x^2 + 2px + 3 = 0.$$

In order to have two solutions (two horizontal tangent lines), we need to have p such that $D > 0$. In this case, $D = 4p^2 - 36$ and

$$D > 0 \Leftrightarrow 4p^2 - 36 > 0$$
$$\Leftrightarrow p^2 > 9$$
$$\Leftrightarrow p < -3 \text{ or } p > 3.$$

Thereby, $y(x)$ has two horizontal tangent lines for any $p < -3$ or $p > 3$.

Exercise 2.34

Since $y(x) = x^4 + 2x + a$, the derivative of $y(x)$ is

$$y'(x) = 4x^3 + 2.$$

Given that the tangent line to be $y = 6x + 7$, we make use of the following system of equations to determine a:

$$\begin{cases} y'(x_0) = 6 \\ y(x_0) = 6x_0 + 7 \end{cases} \Leftrightarrow \begin{cases} 4x_0^3 + 2 = 6 \\ x_0^4 + 2x_0 + a = 6x_0 + 7 \end{cases}$$
$$\Leftrightarrow \begin{cases} x_0 = 1 \\ a = -x_0^4 + 4x_0 + 7 \end{cases}$$
$$\Leftrightarrow \begin{cases} x_0 = 1 \\ a = 10. \end{cases}$$

Exercise 2.35

$y(x) = x^2 \ln(x^4 + 3)$, $\quad y'(x) = 2x \ln(x^4 + 3) + x^2 \cdot \frac{4x^3}{x^4+3}$.
Evaluated at $x = 1$, we have $y'(1) = 2\ln 4 + 1$.

Exercise 2.36

a) Applying the product rule and the chain rule, we obtain

$$y(x) = x^2 e^x,$$
$$y'(x) = 2xe^x + x^2 e^x,$$
$$y'(1) = 2e + e = 3e.$$

b) The tangent line is horizontal when its slope is zero. Hence, we find x such that $y'(x) = 0$:

$$\begin{aligned} y'(x) = 0 &\Leftrightarrow 2xe^x + x^2 e^x = 0 \\ &\Leftrightarrow \underbrace{e^x}_{>0}(2x + x^2) = 0 \\ &\Leftrightarrow 2x + x^2 = 0 \\ &\Leftrightarrow x(2 + x) = 0 \\ &\Leftrightarrow x = 0 \text{ or } x = -2. \end{aligned}$$

Exercise 2.37

Using the chain rule and the product rule, we have

$$y(x) = e^x \ln(x + \frac{1}{2}),$$
$$y'(x) = e^x \ln(x + \frac{1}{2}) + \frac{e^x}{x + \frac{1}{2}},$$
$$y'(0) = \ln(\frac{1}{2}) + 2.$$

Exercise 2.38

a) Applying the chain rule, we have

$$y(x) = \sqrt{x^2+2} = (x^2+2)^{\frac{1}{2}},$$
$$y'(x) = \frac{1}{2}(x^2+2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+2}},$$
$$y'(1) = \frac{1}{\sqrt{3}}.$$

b) The tangent line is horizontal when its slope is zero. Hence, we determine x such that $y'(x) = 0$:

$$y'(x) = 0 \Leftrightarrow \frac{x}{\sqrt{x^2+2}} = 0$$
$$\Leftrightarrow x = 0.$$

Exercise 2.39

a) Using the product rule and the chain rule, we have

$$y(x) = x^2 e^{-x},$$
$$y'(x) = 2xe^{-x} - x^2 e^{-x},$$
$$y'(1) = 2e^{-1} - e^{-1} = e^{-1} = \frac{1}{e}.$$

b) The tangent line is horizontal when its slope is zero. Hence, we find x such that $y'(x) = 0$:

$$y'(x) = 0 \Leftrightarrow 2xe^{-x} - x^2e^{-x} = 0$$
$$\Leftrightarrow \underbrace{e^{-x}}_{>0}(2x - x^2) = 0$$
$$\Leftrightarrow (2x - x^2) = 0$$
$$\Leftrightarrow x(2-x) = 0$$
$$\Leftrightarrow x = 0 \text{ or } x = 2.$$

Exercise 2.40

Applying the chain rule and the product rule, we obtain

$$y(x) = e^{x^2} \ln(2x + \frac{1}{2}),$$
$$y'(x) = 2xe^{x^2} \ln(2x + \frac{1}{2}) + e^{x^2} \cdot \frac{1}{2x + \frac{1}{2}} \cdot 2,$$
$$y'(0) = 4.$$

Exercise 2.41

a) $\epsilon(r) = (d(p) + pd'(p))\dfrac{p}{pd(p)} = 1 + d'(p)\dfrac{p}{d(p)} = 1 + \epsilon(d).$

b) $\epsilon(r) = -2\frac{1}{2}$ implies that $\epsilon(d) = -3\frac{1}{2}$. Then $\%\Delta d \approx \epsilon(d) \cdot \%\Delta p$ gives $\%\Delta d \approx -3\frac{1}{2} \cdot 2 = -7$.

Exercise 2.42

$R(q) = -20q^2 + 1200q$ gives $MR(q) = R'(q) = -40q + 1200$. Hence, $MR(10) = 800$.

Exercise 2.43

a) $1 = y(1 + \Delta x) - y(1) \approx y'(1)\Delta x$.

$y'(x) = \dfrac{4x^3 \cdot e^{3x+1} - x^4 \cdot 3 \cdot e^{3x+1}}{(e^{3x+1})^2} = \dfrac{4x^3 - 3x^4}{e^{3x+1}}$.

Hence, $y'(1) = \dfrac{1}{e^4}$. Hence, $1 \approx \dfrac{1}{e^4}\Delta x$ and $\Delta x \approx e^4$.

b) $\epsilon = y'(1) \cdot \dfrac{1}{y(1)} = \dfrac{1}{e^4} \cdot e^4 = 1$. Hence, $3 \approx 1 \cdot \%\Delta x$, which gives $\%\Delta x \approx 3$.

Exercise 2.44

a) $\%\Delta y \approx \epsilon(y) \cdot \%\Delta x$ gives $-2 \approx \epsilon(y) \cdot 5$. Hence, $\epsilon(y) = -\frac{2}{5}$.

b) $\epsilon(x) = \dfrac{1}{-\frac{2}{5}} = -2\frac{1}{2}$.

c) $8 \approx -\frac{2}{5} \cdot \%\Delta x$ gives $\%\Delta x \approx -20$.

Exercise 2.45

a) $MP(t) = P'(t) = \dfrac{10(1 + e^{-0.1t}) - 10t(-0.1e^{-0.1t})}{(1 + e^{-0.1t})^2} = \dfrac{(10 + t)e^{-0.1t} + 10}{(1 + e^{-0.1t})^2}$.

b) $MP(10) = \frac{20e^{-1}+10}{(1+e^{-1})^2}$.

Exercise 2.46

a)

$$\begin{aligned} y = \frac{1}{x+1} + 1 &\Leftrightarrow y - 1 = \frac{1}{x+1} \\ &\Leftrightarrow x + 1 = \frac{1}{y-1} \\ &\Leftrightarrow x = \frac{1}{y-1} - 1. \end{aligned}$$

Hence, $x(y) = \frac{1}{y-1} - 1 = \frac{1}{y-1} - \frac{y-1}{y-1} = \frac{2-y}{y-1}$, $1 < y < 2$.

b)

$$\begin{aligned} y = \frac{x-20}{x+5} + 4 &\Leftrightarrow (y-4)(x+5) = x - 20 \\ &\Leftrightarrow yx + 5y - 4x - 20 = x - 20 \\ &\Leftrightarrow yx + 5y - 4x = x \\ &\Leftrightarrow yx - 5x = -5y \\ &\Leftrightarrow x(y-5) = -5y \\ &\Leftrightarrow x = \frac{-5y}{y-5}. \end{aligned}$$

Hence, $x(y) = \frac{-5y}{y-5} = \frac{5y}{5-y}$.

Exercise 2.47

a)

$$\begin{aligned} y = 4^{x^3-2} &\Leftrightarrow\ {}^4\log(y) = x^3 - 2 \\ &\Leftrightarrow x^3 = {}^4\log(y) + 2 \\ &\Leftrightarrow x = \sqrt[3]{{}^4\log(y) + 2}. \end{aligned}$$

Hence, $x(y) = \sqrt[3]{{}^4\log(y) + 2}$.Then $x'(y) = \frac{1}{3}({}^4\log(y) + 2)^{-\frac{2}{3}} \cdot \frac{1}{y \ln 4} = \frac{({}^4\log(y)+2)^{-\frac{2}{3}}}{3y\ln(4)}$, $(y > \frac{1}{16})$.

b)

$$\begin{aligned} y = \frac{x+7}{x-2} &\Leftrightarrow y(x-2) = x+7 \\ &\Leftrightarrow yx - 2y = x + 7 \\ &\Leftrightarrow yx - x = 7 + 2y \\ &\Leftrightarrow x(y-1) = 7 + 2y \\ &\Leftrightarrow x = \frac{7+2y}{y-1}. \end{aligned}$$

Hence, $x(y) = \frac{7+2y}{y-1}$. Therefore, $x'(y) = \frac{2(y-1)-(7+2y)}{(y-1)^2} = \frac{-9}{(y-1)^2}$.

Exercise 2.48

a) $y(1) = 2$. $y'(x) = 2007x^{2006} + 1$. $y'(1) = 2008$. Hence, $x'(2) = \frac{1}{2008}$.

b) $y(0) = 1$. $y'(x) = 3x^2 - 4x + 2 + 4^x \ln 4$. $y'(0) = 2 + \ln 4$. Hence, $x'(1) = \frac{1}{\ln(4)+2}$.

c) $y(1) = -\ln 2$. $y'(x) = \frac{x+1}{x} \cdot \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)x}$. $y'(1) = \frac{1}{2}$. Hence, $x'(-\ln 2) = \frac{1}{\frac{1}{2}} = 2$.

d) The inverse is not determined. With the addition $x \geq 0$ we can solve this exercise.

$$\begin{aligned} y = {}^5\log(x^2+3) &\Leftrightarrow 5^y = x^2 + 3 \\ &\Leftrightarrow x^2 = 5^y - 3 \\ &\Leftrightarrow x = \sqrt{5^y - 3}. \end{aligned}$$

Hence, $x(y) = \sqrt{5^y - 3}$, $(y \geq {}^5\log 3)$. Then $x'(y) = \dfrac{1}{2\sqrt{5^y-3}} \cdot 5^y \ln 5$ and $x'(2) = \frac{25\ln(5)}{2\sqrt{22}}$.

Exercise 2.49

a) $\epsilon = Y'(L) \cdot \frac{L}{YL} = (1\frac{1}{2}\sqrt{L} + 1) \cdot \frac{L}{L\sqrt{L}+L} = \frac{1\frac{1}{2}L^{1\frac{1}{2}}+L}{L^{1\frac{1}{2}}+L}$.

b) $\%Y \approx \epsilon \cdot \%\Delta L$, with $\epsilon = 1\frac{4}{9}$ and $\%\Delta Y = 3$ gives $\%\Delta L \approx 2\frac{1}{13}$.

3

Solutions to Chapter 3

Exercise 3.1

a)

$$z(0,0) = 6 - 0 - 0 = 6,$$
$$z(6,0) = 6 - 6 - 0 = 0,$$
$$z(0,6) = 6 - 0 - 6 = 0,$$
$$z(2,4) = 6 - 2 - 4 = 0,$$
$$z(4,2) = 6 - 4 - 2 = 0.$$

b) For example $(x,y) = (5,0)$, $(x,y) = (0,5)$ and $(x,y) = (10,-5)$.

Exercise 3.2

a) Consider the function $z(x_1, x_2) = x_1^2 x_2$. To draw the level curve with function value 2, we set $z = 2$ and solve for variable x_2. The level curve is

$$x_2 = \frac{2}{x_1^2}$$

and is shown in Chapter 7.

b) i) The function is $z(x,y) = x^2 + y$. For an arbitrary level $z = k$, the level curve is given by

$$y = k - x^2.$$

The level curves with different function values k are shown in Chapter 7.

ii) To compute all the function values such that $z(x,y)$ crosses the x-axis twice, we need to guarantee that $y = k - x^2$ has two zeros. Since the discriminant of the equation is $D = 0^2 - 4(-1)(k) = 4k$, we have

$$D > 0 \;\Rightarrow\; k > 0.$$

c) Consider the function $z(x,y) = \dfrac{2xy}{x+y}$. When the function value equals to 4, the level curve becomes

$$\frac{2xy}{x+y} = 4 \;\Leftrightarrow 2xy = 4x + 4y$$
$$\Leftrightarrow xy = 2x + 2y$$
$$\Leftrightarrow xy - 2y = 2x$$
$$\Leftrightarrow y(x-2) = 2x$$
$$\Leftrightarrow y = \frac{2x}{x-2},$$

$x > 2$. See Chapter 7 for the corresponding figure.

Exercise 3.3

Consider the function $4\ln L + 2\ln K = 10$. Solving this for variable K gives

$$\begin{aligned} 4\ln L + 2\ln K = 10 &\Leftrightarrow 2\ln K = 10 - 4\ln L \\ &\Leftrightarrow \ln K = 5 - 2\ln L \\ &\Leftrightarrow e^{\ln K} = e^{5-2\ln L} \\ &\Leftrightarrow K = e^5 e^{-2\ln L} \\ &\Leftrightarrow K = e^5 e^{\ln L^{-2}} \\ &\Leftrightarrow K = e^5 L^{-2} \\ &\Leftrightarrow K = \frac{e^5}{L^2}, \end{aligned}$$

$L > 0$.

Exercise 3.4

a)

$$\begin{aligned} 2 = 2\sqrt[5]{x^2}\sqrt[5]{y^3} &\Leftrightarrow y^{\frac{3}{5}} = \frac{2}{2x^{\frac{2}{5}}} \\ &\Leftrightarrow y = \frac{1}{\sqrt[3]{x^2}}. \end{aligned}$$

if $U = 2$. Similarly, $\frac{2^{\frac{5}{3}}}{\sqrt[3]{x^2}}$ if $U = 4$ and $\frac{3^{\frac{5}{3}}}{\sqrt[3]{x^2}}$ if $U = 6$.

b) See Chapter 7 for the corresponding figure.

Exercise 3.5

a) For example $(x, y) = (10, 10)$, $(x, y) = (20, 10)$, $(x, y) = (100, 10)$, $(x, y) = (10, 50)$, $(x, y) = (10, 1000)$.

b) See Chapter 7 for the corresponding figure.

c) The most preferred ratio is $1 : 1$, as this ratio guarantees that there is no ‘waste’. To see this, consider $U(15, 515) = U(15, 15) = 15$. There is no benefit of having additional 500 units of y, if there is no increase in the units of x. As long as both commodities are costly, it only makes sense to consume them in equal amounts.

Exercise 3.6

a) $U_A(0.5, 0.4) = -0.14 < 0.04 = U_A(0.2, 0.2)$ Hence, Portfolio P_2.

b) $U_B(0.5, 0.4) = 0.18 > 0.12 = U_B(0.2, 0.2)$. Hence, Portfolio P_1.

c) $\mu = U_A + 4\sigma^2$ for investor A, $\mu = U_B + 2\sigma^2$ for investor B. See Chapter 7 for the corresponding figure.

d) $\alpha_A = 8 > 4 = \alpha_B$. Hence, investor A.

Exercise 3.7

a) Solving the budget equation for variable x_2 gives

$$x_2 = -\frac{p_1}{p_2}x_1 + \frac{I}{p_2}.$$

b) Suppose we have strictly positive prices for both goods. To get as much of good 1 as possible, the consumer can spend all his/her income on good 1. Hence, the maximum amount of x_1 is I/p_1. Similarly, the maximum amount of good 2 the consumer can afford is I/p_2.
c) The budget line rotates inwards on the (x_1, x_2)-plane. The slope becomes steeper as the maximum amount of good 1 the consumer can buy decreases.
d) The budget line rotates outwards. The budget line becomes steeper, because the maximum amount of good 2 that the consumer can buy *increases*.
e) If the budget increases, the consumer is able to buy more of *both* goods. Hence, the entire budget line shifts outwards symmetrically, such that the slope remains the same.

Exercise 3.8

The level curve with the function value 1 is given by

$$\begin{aligned} e^{-x+\sqrt{y}} = 1 &\Leftrightarrow \ln(e^{-x+\sqrt{y}}) = \ln 1 \\ &\Leftrightarrow -x + \sqrt{y} = 0 \\ &\Leftrightarrow \sqrt{y} = x \\ &\Leftrightarrow y = x^2, \qquad x \geq 0. \end{aligned}$$

Exercise 3.9

a) Since $U(3,0) = 3a = 9$, we have $a = 3$.
b) The budget restriction is $p_x x + p_y y = x + y = 3$. Thereby, we know that $p_x = p_y = 1$, *i.e.*, both goods are equally expensive and have price 1. Furthermore, with $a = 3$, the consumer gets 3 units of utility for every unit of x he/she consumes, and 2 units of utility for every unit of y he/she consumes. Hence, he/she is always better off consuming x than y. This implies that he/she should spend all his/her income on x, and none on y. According to the budget constraint, he/she will buy 3 units of x and 0 units of y. This gives him/her a utility of 9. Hence, it is indeed the case that $U(3,0) = 9$ is the maximum utility the consumer can achieve.

Exercise 3.10

a) The isoquant is $\min\{2x_1, x_2\} = 12$. We know that the level curve of the minimum function is always L-shaped. Hence, to draw the level curve, we only need to determine the corner point of the L-shaped curve, which is given by

$$\begin{aligned} 12 &= 2x_1 &\Rightarrow\quad x_1 &= 6, \\ 12 &= x_2 &\Rightarrow\quad x_2 &= 12. \end{aligned}$$

See Chapter 7 for the corresponding figure.
b) The isoquant is $\min\{x_1, 3x_2\} = 12$. The corner point of the L-shaped curve is given by

$$\begin{aligned} 12 &= x_1 &\Rightarrow\quad x_1 &= 12, \\ 12 &= 3x_2 &\Rightarrow\quad x_2 &= 4. \end{aligned}$$

See Chapter 7 for the corresponding figure.
c) As long as the costs for both input factors are strictly positive, the cost-minimizing input combination is in the corner point. Therefore, if $a = 2$ and $b = 1$, we require $x_1 = 6$ and $x_2 = 12$. At prices $p_1 = 2$ and $p_2 = 3$, this gives costs of production of $6 \cdot 2 + 12 \cdot 3 = 48$. If $a = 1$ and $b = 3$, we need $x_1 = 12$ and $x_2 = 4$. At prices $p_1 = 2$ and $p_2 = 3$, the costs of production are $12 \cdot 2 + 4 \cdot 3 = 36$.

Exercise 3.11

a) $3 = 9LK^2 \Leftrightarrow K^2 = \frac{3}{9L} \Leftrightarrow K = \sqrt{\frac{1}{3L}}$. See Chapter 7 for the corresponding figure.

b) $6 = 9L \cdot 2^2 \Leftrightarrow L = \frac{1}{6}$.

Exercise 3.12

a) $x = 8$ and $y \geq 3$, or $y = 3$ and $x \geq 8$. See Chapter 7 for the corresponding figure.

b) $k = \min\{9, y^2\} \Leftrightarrow k = y^2 \Leftrightarrow y = \sqrt{k}$.

c) With $x = 8$ it is not possible to have $k > 9$, but $k = 9$ for all $y \geq 3$.

Exercise 3.13

a) $4 - \frac{1}{2} \cdot 0.5^2 = 5 - \frac{1}{2}a^2 \Leftrightarrow 3\frac{7}{8} = 5 - \frac{1}{2}a^2 \Leftrightarrow a = 1\frac{1}{2}$.

b) $4 - \frac{1}{2} \cdot 0.5^2 > b - \frac{1}{2} \cdot 0.25^2 \Leftrightarrow b < 3\frac{29}{32}$.

4
Solutions to Chapter 4

Exercise 4.1

a) $z'_x(x,y) = 2y - 10x, \quad z'_y(x,y) = 2x + 1.$

b) $z'_x(x,y) = 6(1-x^2y)^5(-2xy) = -12xy(1-x^2y)^5,$
$z'_y(x,y) = 6(1-x^2y)^5(-x^2) = -6x^2(1-x^2y)^5.$

c)

$$z'_x(x,y) = \frac{\frac{1}{2}(xy)^{-\frac{1}{2}}y(x+y) - (xy)^{\frac{1}{2}}}{(x+y)^2} = \frac{y(y-x)}{2\sqrt{xy}(x+y)^2},$$

$$z'_y(x,y) = \frac{\frac{1}{2}(xy)^{-\frac{1}{2}}x(x+y) - (xy)^{\frac{1}{2}}}{(x+y)^2} = \frac{x(x-y)}{2\sqrt{xy}(x+y)^2}.$$

Exercise 4.2

a) $z'_x(x,y) = y(1-y^2)^{\frac{1}{2}}, \quad z'_x(2,0) = 0,$

$z'_y(x,y) = x(1-y^2)^{\frac{1}{2}} + xy \cdot \frac{1}{2}(1-y^2)^{-\frac{1}{2}}(-2y), \quad z'_y(2,0) = 2.$

b) $Q'_L(L,K) = 0.1L^{-\frac{1}{2}}K^{\frac{1}{2}}, \quad Q'_L(4,16) = 0.2,$

$Q'_K(L,K) = 0.1L^{\frac{1}{2}}K^{-\frac{1}{2}}, \quad Q'_K(4,16) = 0.05.$

Exercise 4.3

a) $U'_1(x_1,x_2) = 30x_1^4x_2^4$, $U'_2(x_1,x_2) = 24x_1^5x_2^3$.

b) $U'_1(x_1,x_2) = 6(\sqrt{x_1}+\sqrt{x_2}) \cdot \frac{1}{2\sqrt{x_1}} = \frac{3(\sqrt{x_1}+\sqrt{x_2})}{\sqrt{x_1}},$
$U'_2(x_1,x_2) = 6(\sqrt{x_1}+\sqrt{x_2}) \cdot \frac{1}{2\sqrt{x_2}} = \frac{3(\sqrt{x_1}+\sqrt{x_2})}{\sqrt{x_2}}.$

c) $U'_x(x,y) = y + 10$, $U'_y(x,y) = x + 4$.

d) $U'_x(x,y) = \frac{4}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}}$, $U'_y(x,y) = \frac{8}{3}x^{\frac{1}{3}}y^{-\frac{1}{3}}$.

Exercise 4.4

a) Since $z(x,y) = kx^{\alpha}y^{1-\alpha}$, the partial derivatives are

$$\frac{\partial z(x,y)}{\partial x} = k\alpha x^{\alpha-1}y^{1-\alpha} = \frac{\alpha}{x}kx^{\alpha}y^{1-\alpha} = \frac{\alpha}{x}z(x,y)$$

and

$$\frac{\partial z(x,y)}{\partial y} = k(1-\alpha)x^{\alpha}y^{-\alpha} = \frac{1-\alpha}{y}kx^{\alpha}y^{1-\alpha} = \frac{1-\alpha}{y}z(x,y).$$

b) Using the results from part (a), we have

$$\begin{aligned} x \cdot \frac{\partial z(x,y)}{\partial x} + y \cdot \frac{\partial z(x,y)}{\partial y} &= x \cdot \frac{\alpha}{x} z(x,y) + y \cdot \frac{1-\alpha}{y} z(x,y) \\ &= \alpha \cdot z(x,y) + (1-\alpha) \cdot z(x,y) \\ &= z(x,y). \end{aligned}$$

Exercise 4.5

a)

$$\begin{aligned} u(x,16) &= \min\{2x,16\}, \\ u(6,16) &= \min\{12,16\} = 12, \\ u(10,16) &= \min\{20,16\} = 16. \end{aligned}$$

b) Since $2x < y$ at $(6,16)$, $u(x,y)$ behaves like $2x$. Hence, $u'_x(6,16) = 2$. Similarly, since $y < 2x$ at $(10,16)$, $u(x,y)$ behaves like y. Hence, $u'_x(10,16) = 0$.
c) Since $2x < y$ at $(8,20)$, $u(x,y)$ behaves like $2x$. Hence, $u'_x(8,20) = 2$.
d) Since $y < 2x$ at $(10,16)$, $u(x,y)$ behaves like y. Hence, $u'_y(10,16) = 1$. Similarly, since $2x < y$ at $(8,20)$, $u(x,y)$ behaves like $2x$. Hence, $u'_y(8,20) = 0$.

Exercise 4.6

a) $u'_x(x,y,z) = x^{-\frac{2}{3}} y^{\frac{1}{6}} z^{\frac{1}{2}}$, $u'_y(x,y,z) = \frac{1}{2} x^{\frac{1}{3}} y^{-\frac{5}{6}} z^{\frac{1}{2}}$, $u'_z(x,y,z) = 1\frac{1}{2} x^{\frac{1}{3}} y^{\frac{1}{6}} z^{-\frac{1}{2}}$.
b) $u'_x(x,y,z) = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $u'_y(x,y,z) = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $u'_z(x,y,z) = \frac{z}{\sqrt{x^2+y^2+z^2}}$.

Exercise 4.7

a)

$$\Delta z \approx z'_x(4,1)\Delta x + z'_y(4,1)\Delta y = \tfrac{1}{4} \cdot (-0.2) + 1 \cdot 0.01 = -0.04.$$

b)

$$\Delta z \approx z'_x(4,1)\Delta x + z'_y(4,1)\Delta y = \tfrac{1}{4} \cdot 0.04 + 1 \cdot (-0.01) = 0.$$

Exercise 4.8

a) Since $MPP_L(L,K) = F'_L(L,K) = 3L^{-\frac{2}{5}} K^{\frac{2}{5}}$, the marginal product of labor at $(L,K) = (200,400)$ is $MPP_L(200,400) = 3(200)^{-\frac{2}{5}}(400)^{\frac{2}{5}} = 3 \cdot 2^{\frac{2}{5}} = 3 \cdot 4^{\frac{1}{5}} = 3\sqrt[5]{4}$.
b)

$$\begin{aligned} \Delta F &\approx F'_L(200,400)\Delta L + F'_K(200,400)\Delta K \\ &\Leftrightarrow\ 1 \approx 3\sqrt[5]{4}\Delta L + F'_K(200,400) \cdot 0 \\ &\Leftrightarrow\ \Delta L \approx \frac{1}{3\sqrt[5]{4}}. \end{aligned}$$

c) The marginal product of capital is $F'_K(L,K) = 2L^{\frac{3}{5}} K^{-\frac{3}{5}}$. At $(L,K) = (200,400)$,

$F'_K(200,400) = 2(200)^{\frac{3}{5}}(400)^{-\frac{3}{5}} = 2(\frac{1}{2})^{\frac{3}{5}} = \frac{2}{\sqrt[5]{8}}$. Hence, we have

$$\begin{aligned}\Delta F &\approx F'_L(200,400)\Delta L + F'_K(200,400)\Delta K\\ &\Leftrightarrow\ 2 \approx 3\sqrt[5]{4}\cdot 0 + \frac{2}{\sqrt[5]{8}}\cdot \Delta K\\ &\Leftrightarrow\ \Delta K \approx \frac{2}{\frac{2}{\sqrt[5]{8}}} = \sqrt[5]{8}.\end{aligned}$$

Exercise 4.9

By Exercise 4.4 we know that $z'_x(x,y) = \frac{\alpha}{x}\cdot z(x,y)$. Hence, $\epsilon_x = \frac{\alpha}{x}\cdot z(x,y)\cdot \frac{x}{z(x,y)} = \alpha$. Similarly, $z'_y(x,y) = \frac{1-\alpha}{y}\cdot z(x,y)$. Hence, $\epsilon_y = \frac{1-\alpha}{y}\cdot z(x,y)\cdot \frac{y}{z(x,y)} = 1-\alpha$.

Exercise 4.10

a) Since $Q'_L(L,K) = \frac{1}{6}L^{-\frac{2}{3}}K^{\frac{2}{3}}$, the elasticity of output with respect to labor is then

$$\epsilon_L = Q'_L(L,K)\cdot\frac{L}{Q(L,K)} = \tfrac{1}{6}L^{-\frac{2}{3}}K^{\frac{2}{3}}\cdot\frac{L}{\frac{1}{2}L^{\frac{1}{3}}K^{\frac{2}{3}}} = \frac{1}{3}.$$

Hence,

$$\%\Delta Q \approx \epsilon_L\cdot\%\Delta L = \tfrac{1}{3}\cdot 1 = \tfrac{1}{3}.$$

b) The output increases approximately by

$$\%\Delta Q \approx \epsilon_L\cdot\%\Delta L = \tfrac{1}{3}\cdot 3 = 1.$$

c)

$$\%\Delta Q \approx \epsilon_L\cdot\%\Delta L \ \Leftrightarrow\ 2 \approx \tfrac{1}{3}\%\Delta L \ \Leftrightarrow\ \%\Delta L \approx 6.$$

Exercise 4.11

$x(5,12,400) = 457\frac{2}{3}$. $\epsilon_{p_x} = x'(p_x,p_y,I)\cdot\frac{p_x}{x(p_x,p_y,I)} = -3\cdot\frac{5}{457\frac{2}{3}} = -\frac{45}{1373}$.

Similarly, $\epsilon_{p_y} = \frac{18}{1373}$, $\epsilon_I = \frac{800}{1373}$.

Exercise 4.12

a) Consider the composite function $Z(t) = z(x(t),y(t))$, where

$$z(x,y) = x^4y^3,\quad x(t) = \sqrt{t}\quad\text{and}\quad y(t) = 3t-4.$$

To determine the derivative of $Z(t)$, we need $z'_x(x,y)$, $z'_y(x,y)$, $x'(t)$ and $y'(t)$:

$$\begin{aligned}&x'(t) = \frac{1}{2\sqrt{t}},\qquad y'(t) = 3,\\ &z'_x(x,y) = 4x^3y^3,\qquad z'_y(x,y) = 3x^4y^2.\end{aligned}$$

Evaluated at $t=4$, we have

$$\begin{aligned}&x(4) = 2,\qquad y(4) = 8,\\ &x'(4) = \frac{1}{4},\qquad y'(4) = 3,\\ &z'_x(x(4),y(4)) = z'_x(2,8) = 4(2^3)(8^3) = 16384,\\ &z'_y(x(4),y(4)) = z'_y(2,8) = 3(2^4)(8^2) = 3072.\end{aligned}$$

According to the chain rule of a composite function, we finally get

$$\begin{aligned} Z'(4) &= z'_x(x(4), y(4)) \cdot x'(4) + z'_y(x(4), y(4)) \cdot y'(4) \\ &= 16384 \cdot \frac{1}{4} + 3072 \cdot 3 \\ &= 13312. \end{aligned}$$

b) $Z(t) = (\sqrt{t})^4 (3t-4)^3 = t^2(3t-4)^3$.

$$\begin{aligned} Z'(t) &= 2t(3t-4)^3 + t^2 \cdot 3(3t-4)^2 \cdot 3 \\ &= 2t(3t-4)^3 + 9(3t-4)^2. \end{aligned}$$

$Z'(4) = 2 \cdot 1(3 \cdot 1 - 4)^3 + 9(3 \cdot 1 - 4)^2 = 13312.$

Exercise 4.13

a) Consider the composite function $Z(x) = z(x, y(x))$, where

$$y(x) = \sqrt{5 - x^2} \quad \text{and} \quad z(x, y) = 3x^{\frac{1}{3}} y^{\frac{2}{3}}.$$

To determine the derivative of $Z(x)$, we need $z'_x(x, y)$, $z'_y(x, y)$ and $y'(x)$:

$$\begin{aligned} &y'(x) = -\frac{x}{\sqrt{5 - x^2}}, \\ &z'_x(x, y) = x^{-\frac{2}{3}} y^{\frac{2}{3}}, \qquad z'_y(x, y) = 2x^{\frac{1}{3}} y^{-\frac{1}{3}}. \end{aligned}$$

Evaluated at $x = 1$, we have

$$\begin{aligned} &y(1) = 2, \qquad y'(1) = -\frac{1}{2} = -2^{-1}, \\ &z'_x(1, y(1)) = z'_x(1, 2) = 1^{-\frac{2}{3}} 2^{\frac{2}{3}} = 2^{\frac{2}{3}}, \\ &z'_y(1, y(1)) = z'_y(1, 2) = 2(1^{\frac{1}{3}}) 2^{-\frac{1}{3}} = 2^{\frac{2}{3}}. \end{aligned}$$

According to the chain rule of a composite function, we finally obtain

$$\begin{aligned} Z'(1) &= z'_x(1, y(1)) + z'_y(1, y(1)) \cdot y'(1) \\ &= 2^{\frac{2}{3}} + 2^{\frac{2}{3}} \cdot (-2^{-1}) = 2^{-\frac{1}{3}}. \end{aligned}$$

b) $Z(x) = 3x^{\frac{1}{3}} \cdot (\sqrt{5 - x^2})^{\frac{2}{3}} = 3x^{\frac{1}{3}} \cdot (5 - x^2)^{\frac{1}{3}} = 3(5x - x^3)^{\frac{1}{3}}$.

$$Z'(x) = 3 \cdot \tfrac{1}{3}(5x - x^3)^{-\frac{2}{3}} \cdot (5 - 3x^2).$$

$Z'(1) = (5 \cdot 1 - 1^3)^{-\frac{2}{3}} \cdot (5 - 3 \cdot 1^2) = 4^{-\frac{2}{3}} \cdot 2 = 2^{-\frac{1}{3}}.$

Exercise 4.14

In $(L, K) = (4, 6)$, the level is

$$F(4, 16) = 20.$$

Solving the level curve with level 20 for K gives

$$3L + 2\sqrt{K} = 20 \;\Leftrightarrow\; K(L) = (10 - \tfrac{3}{2}L)^2.$$

Then differentiate to find the slope at $L = 4$

$$K'(L) = -2\left(10 - \tfrac{3}{2}L\right) \cdot \tfrac{3}{2} \qquad \Rightarrow \qquad K'(4) = -12$$

Exercise 4.15

a) Since $U_1'(x_1, x_2) = 30x_1^4 x_2^4$ and $U_2'(x_1, x_2) = 24x_1^5 x_2^3$ (see Exercise 4.3), we obtain

$$slope = -\frac{30x_1^4 x_2^4}{24x_1^5 x_2^3} = -\frac{5x_2}{4x_1}.$$

Substituting $(x_1, x_2) = (2, 1)$, we obtain

$$slope = -\frac{5}{8}.$$

b) Similarly, we have

$$slope = -\frac{U_1'(x_1, x_2)}{U_2'(x_1, x_2)} = -\frac{6(\sqrt{x_1} + \sqrt{x_2})x_1^{-\frac{1}{2}}/2}{6(\sqrt{x_1} + \sqrt{x_2})x_2^{-\frac{1}{2}}/2} = -\frac{x_2^{\frac{1}{2}}}{x_1^{\frac{1}{2}}}.$$

And, the slope at $(4, 9)$ is

$$slope = -\frac{3}{2}.$$

c) Similarly,

$$slope = -\frac{U_x'(x, y)}{U_y'(x, y)} = -\frac{y + 10}{x + 4}$$

The slope at $(1, 2)$ is given by

$$slope = -\frac{12}{5}$$

d) The partial derivatives are $U_x'(x, y) = \frac{4}{3}x^{-\frac{2}{3}}y^{\frac{2}{3}}$ and $U_y'(x, y) = \frac{8}{3}x^{\frac{1}{3}}y^{-\frac{1}{3}}$. Hence, the slope s of the tangent line to the level curve in $(9, 27)$ is equal to

$$slope = -\frac{U_x'(9, 27)}{U_y'(9, 27)} = -\frac{27}{2 \cdot 9} = -1\tfrac{1}{2}.$$

Exercise 4.16

If two curves touch each other in the point (x_0, y_0), then we have a common tangent line to both curves. This means that the slopes of the tangent lines to both curves are the same in (x_0, y_0), and (x_0, y_0) is positioned on both curves.

First, we can immediately see that $(1, 1)$ satisfies the equation $x^2 + xy + y^2 = 3$. So, $(1, 1)$ is on the curve $x^2 + xy + y^2 = 3$ and the level curve of $z(x, y)$ in $(1, 1)$.

Next, we show that the slopes of the tangent lines to the equation and to the level curve are the same in $(1, 1)$. By use of the partial derivatives of $z(x, y)$, $z_x'(x, y) = 2x$ and $z_y'(x, y) = 2y$, the slope of the tangent line to the level curve of $z(x, y)$ in $(1, 1)$ equals

$$-\frac{z_x'(1, 1)}{z_y'(1, 1)} = -\frac{2 \cdot 1}{2 \cdot 1} = -1.$$

To calculate the slope of the tangent line to $x^2+xy+y^2=3$, we also use the partial derivatives. Take $g(x,y)=x^2+xy+y^2$, then $g'_x(x,y)=2x+y$ and $g'_y(x,y)=x+2y$. Hence, the slope is

$$-\frac{g'_x(1,1)}{g'_y(1,1)}=-\frac{2\cdot 1+1}{1+2\cdot 1}=-1,$$

which is the same as the slope of the tangent line to the level curve of $z(x,y)$ in $(1,1)$. Using the above two facts, the proof is completed.

Exercise 4.17

a) In Exercise 4.15 we obtained that the slope of the tangent line of the level curve in $(x_1,x_2)=(2,1)$ is $-\frac{5}{8}$. Hence, $MRS(2,1)=\frac{5}{8}$.
b) In Exercise 4.15 we obtained that the slope of the tangent line of the level curve in $(x_1,x_2)=(4,9)$ is $-\frac{3}{2}$. Hence, $MRS(4,9)=\frac{3}{2}$.
c) In Exercise 4.15 we obtained that the slope of the tangent line of the level curve in $(x,y)=(1,2)$ is $-\frac{12}{5}$. Hence, $MRS(1,2)=\frac{12}{5}$.
d) In Exercise 4.15 we obtained that the slope of the tangent line of the level curve in $(x,y)=(9,27)$ is $-\frac{3}{2}$. Hence, $MRS(9,27)=\frac{3}{2}$.

Exercise 4.18

a) i) Since $F'_x(x,y)=p_1$ and $F'_y(x,y)=p_2$, the MRTS can be written as

$$MRTS(x,y)=\frac{p_1}{p_2}.$$

ii) The isoquant (level curve) with a production level equals to k is given by

$$\begin{aligned}p_1x+p_2y=k\Rightarrow p_2y&=k-p_1x\\ \Rightarrow y&=\frac{k}{p_2}-\frac{p_1}{p_2}x,\end{aligned}$$

where the slope is $-\frac{p_1}{p_2}$. Given p_1 and p_2 fixed, the isoquants for different production levels k are presented in Chapter 7.

b) i) We have the production function $F(x,y)=\min\{2x,5y\}$. The isoquants are the L-shaped lines. The isoquants for different production levels k are presented in Chapter 7.

ii) $2\cdot 7>5\cdot 2$, which means that the function behaves as $5K$. Hence,

$$MRTS(7,2)=\frac{F_L(7,2)}{F_K(7,2)}=\frac{0}{5}=0$$

$2\cdot 5<5\cdot 4$, which means that the function behaves as $2L$. Hence,

$$MRTS(5,4)=\frac{F_L(5,4)}{F_K(5,4)}=\frac{2}{0}=\infty$$

Intuitively, since the two factors are perfect complements, the marginal product of the scarce factor is always infinity, because we can immediately produce more total output, while the marginal product of the abundant factor is always zero, as any additional unit of this factor does not increase output.

Exercise 4.19

Since $z(x,y) = xe^{xy} + \ln(x+y^2)$, the partial derivative of $z(x,y)$ with respect to x is

$$z'_x(x,y) = e^{xy} + xye^{xy} + \frac{1}{x+y^2}.$$

Evaluated at $(x,y) = (2,1)$, we have

$$z'_x(2,1) = e^2 + 2e^2 + 3^{-1} = 3e^2 + \tfrac{1}{3}.$$

Exercise 4.20

a) $z'_y(x,y) = \ln(2x-y^2) + y \cdot \frac{-2y}{2x-y^2}$. Hence, $z'_y(3,2) = -4 + \ln 2$.

b) $z'_y(x,y) = x^2 + \frac{x}{(1+xy)\ln 3} + 2^{xy} x \ln 2$. Hence, $z'_y(1,0) = 1 + \frac{1}{\ln(3)} + \ln 2$.

c) $z'_y(x,y) = \frac{2y(x^2+2y)-2y^2}{(x^2+2y)^2}$. Hence, $z'_y(1,2) = \frac{12}{25}$.

Exercise 4.21

a) The partial derivatives are

$$X'_K(L,K) = \frac{4(2K^{\frac{1}{2}} + L^{\frac{2}{3}})^3}{K^{\frac{1}{2}}}$$

and

$$X'_L(L,K) = \frac{8(2K^{\frac{1}{2}} + L^{\frac{2}{3}})^3}{3L^{\frac{1}{3}}}$$

b) Then the slope s in point $(8,4)$ is given by

$$s = -\frac{X'_L(8,4)}{X'_K(8,4)} = -\frac{2}{3}.$$

c) By definition, it holds that $MRTS(8,4) = -s$. Hence

$$MRTS(8,4) = \frac{2}{3}.$$

Exercise 4.22

Consider the composite function $Z(t) = z(x(t), y(t))$, where $x(t) = \ln t$ and $y(t) = e^{2t}$. To determine the derivative of $Z(t)$, we need $z'_x(x,y)$, $z'_y(x,y)$, $x'(t)$ and $y'(t)$:

$$x'(t) = \frac{1}{t}, \qquad y'(t) = 2e^{2t},$$
$$z'_x(x,y) = 6xy + y, \qquad z'_y(x,y) = 3x^2 + x.$$

Evaluated at $t = 4$, we have

$$x(4) = \ln 4, \qquad y(4) = e^8,$$
$$x'(4) = \frac{1}{4}, \qquad y'(4) = 2e^8,$$
$$z'_x(x(4), y(4)) = z'_x(\ln 4, e^8) = 6e^8 \ln 4 + e^8 = e^8(6\ln 4 + 1),$$
$$z'_y(x(4), y(4)) = z'_y(\ln 4, e^8) = 3(\ln 4)^2 + \ln 4 = \ln 4(3\ln 4 + 1).$$

According to the chain rule of a composite function, we finally obtain

$$\begin{aligned}
Z'(4) &= z'_x(x(4), y(4)) \cdot x'(4) + z'_y(x(4), y(4)) \cdot y'(4) \\
&= e^8(6\ln 4 + 1) \cdot \frac{1}{4} + \ln 4(3\ln 4 + 1)2e^8 \\
&= e^8(\frac{1}{4} + \ln(4)(3\frac{1}{2} + 6\ln(4))).
\end{aligned}$$

Exercise 4.23

Consider the composite function $Z(x) = z(x, y(x))$, where $y(x) = 4x + \sqrt{x}$ and $z(x, y) = \ln(x^2 + y^5)$. To determine the derivative of $Z(x)$, we need $z'_x(x, y)$, $z'_y(x, y)$ and $y'(x)$:

$$y'(x) = 4 + \frac{1}{2\sqrt{x}}, \qquad z'_x(x, y) = \frac{2x}{x^2 + y^5}, \qquad z'_y(x, y) = \frac{5y^4}{x^2 + y^5}.$$

Evaluated at $x = 1$, we have

$$\begin{aligned}
&y(1) = 5, \qquad y'(1) = \frac{9}{2}, \\
&z'_x(1, y(1)) = z'_x(1, 5) = \frac{2}{1 + 5^5}, \\
&z'_y(1, y(1)) = z'_y(1, 5) = \frac{5^5}{1 + 5^5}.
\end{aligned}$$

According to the chain rule of a composite function, we finally get

$$\begin{aligned}
Z'(1) &= z'_x(1, y(1)) + z'_y(1, y(1)) \cdot y'(1) \\
&= \frac{2}{1 + 5^5} + \frac{5^5}{1 + 5^5} \cdot \frac{9}{2} \\
&= 4\tfrac{3121}{6252}.
\end{aligned}$$

Exercise 4.24

a) $1^2 = 1 < 2 = 1 + 1$. Hence, $z'_x(1, 1) = 2 \cdot 1 = 2$ and $z'_y(1, 1) = 0$.

b) $MRS(1, 1) = \frac{z'_x(1,1)}{z'_y(1,1)} = \frac{2}{0}(= \infty)$.

c) This is a vertical line going through $(x, y) = (1, 1)$. Hence, $x = 1$.

Exercise 4.25

$Q'_L(L, K) = 2LK$ and hence, $Q'_L(3, 2) = 12$. Further, $Q'_K(L, K) = L^2 + 2K$ and hence, $Q'_K(3, 2) = 13$. Then $\Delta Q \approx Q'_K(L, K) \cdot \Delta K + Q'_L(L, K)\Delta L$ gives $\Delta Q \approx 13 \cdot 2 + 12 \cdot (-1) = 14$.

Exercise 4.26

Since $U(x, y)$ is a Cobb-Douglas function, $\epsilon_x = \frac{1}{3}$ (See Exercise 4.9). $\%\Delta x \approx \frac{\%\Delta U}{\epsilon_x} = \frac{2}{\frac{1}{3}} = 6$.

Exercise 4.27

a) $MRS(x, y) = \frac{U'_x(x,y)}{U'_y(x,y)} = \frac{2xy^3}{3x^2y^2}$ and hence, $MRS(1, 1) = \frac{2}{3}$. Hence, the slope is given by $-\frac{2}{3}$. Hence, $y = -\frac{2}{3}x + b$ and since $1 = -\frac{2}{3} \cdot 1 + b$ it holds that $b = 1\frac{2}{3}$ and therefore, $y = -\frac{2}{3}x + 1\frac{2}{3}$.

b) $MRS(10, 20) = 1\frac{1}{3}$. $\Delta y \approx -MRS(x, y) \cdot \Delta x$. Hence, $\Delta x \approx -\frac{1}{1\frac{1}{3}} = -\frac{3}{4}$.

Exercise 4.28

a) Since $Y(L,K)$ is a Cobb-Douglas function, $\epsilon_L = \frac{1}{5}$ (See Exercise 4.9).

b) $\%\Delta Y \approx \epsilon_L \cdot \%\Delta L = \frac{1}{5} \cdot 2 = \frac{2}{5}$.

Exercise 4.29

$Z(t) = (2t+1)^2(4t+1)$ and hence, $Z'(t) = 2 \cdot 2(2t+1)(4t+1) + (2t+1)^2 \cdot 4 = 48t^2 + 40t + 8$. Using the abc-formula on $Z'(t) = 0$ we find the zeros of $Z'(t)$: $t = -\frac{1}{2}$ and $t = -\frac{1}{3}$.

Exercise 4.30

a) $z'_x(x,y) = 2y^2 \cdot e^{4-2x} \cdot (-2)$ and hence, $z'_x(2,2) = -16$. Further, $z'_y(x,y) = 4y \cdot e^{4-2x}$ and hence, $z'_y(2,2) = 8$. Then $\Delta z \approx z'_x(2,2)\Delta x + z'_y(2,2)\Delta y$ gives $-2 \approx -16 \cdot \Delta x + 8 \cdot (-3)$ and hence, $\Delta x \approx -\frac{22}{16}$.

b) $\epsilon_y = 4y \cdot e^{4-2x} \cdot \frac{y}{2y^2 e^{4-2x}}$ and hence, $\epsilon_y = 2$. Hence, $\%\Delta y \approx \frac{\%\Delta z}{\epsilon_y} = 1\frac{1}{2}$.

Exercise 4.31

a) $U(2,1) = 0$ and hence, $0 = xy + x - 4 \Leftrightarrow xy = 4 - x \Leftrightarrow y = \frac{4-x}{x}$. Therefore, $y(x) = \frac{4-x}{x}$.

b) $MRS(x,y) = \frac{U'_x(x,y)}{U'_y(x,y)} = \frac{y+1}{x}$. Then $slope = -MRS(1,2) = -3$. Therefore, $y = -3x + b$. Since, $2 = -3 \cdot 1 + b$ we obtain $b = 5$, which gives $y = -3x + 5$.

Exercise 4.32

$\Delta z \approx z'_x(1,1)\Delta x$ gives $-2 \approx z'_x(x,y) \cdot 3$. Therefore, $z'_x(1,1) = -\frac{2}{3}$.
Since $\epsilon_x = z'_x(x,y) \cdot \frac{x}{z(x,y)} = -\frac{2}{3} \cdot \frac{1}{5}$ we obtain $\epsilon_x = -\frac{2}{15}$.

Exercise 4.33

a) $MPP_L(L,K) = Q'_L(L,K) = 4L^{-\frac{3}{5}}K^{\frac{3}{5}}$, $MPP_K(L,K) = Q'_K(L,K) = 6L^{\frac{2}{5}}K^{-\frac{2}{5}}$.

b) $MRTS(L,K) = \dfrac{MPP_L(L,K)}{MPP_K(L,K)} = \dfrac{2K}{3L}$.

Exercise 4.34

$Z(x) = 10e^{(x^2+\sqrt{x})^2 \cdot \sqrt{x}} = 10e^{x^{4\frac{1}{2}} + x^{1\frac{1}{2}} + 2x^3}$.
Hence $Z'(x) = 10e^{x^{4\frac{1}{2}} + x^{1\frac{1}{2}} + 2x^3} \cdot (4\frac{1}{2}x^{3\frac{1}{2}} + 1\frac{1}{2}x^{\frac{1}{2}} + 6x^2)$.
Hence, $Z'(1) = 120e^4$.

Exercise 4.35

$3y = 16 - x$ gives $y = \frac{16}{3} - \frac{1}{3}x$. Hence, $MRS(x,y) = -slope$ gives $\frac{y}{x+2y} = \frac{1}{3}$, which results in $y = x$. Therefore, $y = \frac{16}{3} - \frac{1}{3}y$, which gives $y = 4$. Consequently, $x = 4$. Hence, $a = 4$ and $b = 4$.

Exercise 4.36

$ax + by = 18$ gives $y = \frac{18}{b} - \frac{a}{b}x$. Hence, $MRS(x,y) = -slope$ gives $\frac{\sqrt{y}}{x \cdot \frac{1}{2\sqrt{y}}} = \frac{1}{2} = \frac{a}{b}$. Hence, $2a = b$. Therefore, $1 = \frac{18}{b} - \frac{1}{2} \cdot 4$, which gives $b = 6$ and therefore $a = 3$. Hence, $a = 3$, $b = 6$.

Exercise 4.37

Since the level curve is tangent to the x-axis, $b = 0$. Moreover, $MRS(x,y) = -slope$ gives $\frac{2x+6y+3}{6x+4y} = 0$, which gives (since $y = 0$) $2x + 3 = 0$. Hence, $a = -1\frac{1}{2}$, $b = 0$.

Exercise 4.38

First, we compute the partial derivatives. These are symmetrical (since the original function is symmetrical) and given by

$$U'_x(x,y) = \frac{(x^{\frac{1}{3}} + y^{\frac{1}{3}})^2}{x^{\frac{2}{3}}} \qquad \text{and} \qquad U'_y(x,y) = \frac{(x^{\frac{1}{3}} + y^{\frac{1}{3}})^2}{y^{\frac{2}{3}}}$$

Then we know that the slope s of the tangent line is

$$s = -\frac{\frac{(x^{\frac{1}{3}}+y^{\frac{1}{3}})^2}{x^{\frac{2}{3}}}}{\frac{(x^{\frac{1}{3}}+y^{\frac{1}{3}})^2}{y^{\frac{2}{3}}}} = -\left(\frac{y}{x}\right)^{\frac{2}{3}}$$

Since we are trying to find the point such that the slope is equal to -4, we need

$$\left(\frac{y}{x}\right)^{\frac{2}{3}} = 4$$

which we can rearrange to give

$$y = 8x$$

Furthermore, we are asked to find the point such that $U(x,y) = 54$. Since $y = 8x$, we therefore need

$$U(x,8x) = \left(x^{\frac{1}{3}} + (8x)^{\frac{1}{3}}\right)^3 = (3x^{\frac{1}{3}})^3 = 54.$$

Hence, we need to have $x = 2$, which implies $y = 16$. Our solution is therefore $(2,16)$.

Exercise 4.39

First, compute the MRS as

$$MRS(x,y) = \frac{6x^2y^4}{8x^3y^3} = \frac{3}{4}\frac{y}{x}$$

which we need to be equal to 3. Hence, we must have

$$y = 4x$$

Now that we know the ratio, go about finding x such that $U(x,4x) = 512$:

$$U(x,4x) = 2x^3 4^4 x^4 = 512x^7 = 512 \qquad \Rightarrow x^* = 1, \qquad y^* = 4$$

Exercise 4.40

$U'_x(x,y) = aye^{ax}$ and $U'_y(x,y) = e^{ax}$. Hence, the slope is given by

$$s = -\frac{U'_x(1,3)}{U'_y(1,3)} = -3 \cdot a \qquad \Rightarrow 3a = 3$$

Hence, $a = 1$. Then the utility level is $U(1,3) = 3e$.

5

Solutions to Chapter 5

Exercise 5.1

a) Since $y(x) = 2x^3 - 9x^2 - 12x$, the first order derivative is

$$y'(x) = 6x^2 - 18x - 12 = 6(x^2 - 3x - 2).$$

The zeros of $y'(x)$ are given by

$$x = \frac{3 - \sqrt{17}}{2} \text{ and } x = \frac{3 + \sqrt{17}}{2}.$$

From the sign survey of $y'(x)$, we have the following.

$$\begin{aligned} &\text{If } x \in (-\infty, \frac{3 - \sqrt{17}}{2}], && y'(x) \geq 0. \\ &\text{If } x \in [\frac{3 - \sqrt{17}}{2}, \frac{3 + \sqrt{17}}{2}], && y'(x) \leq 0. \\ &\text{If } x \in [\frac{3 + \sqrt{17}}{2}, \infty), && y'(x) \geq 0. \end{aligned}$$

So, the function is increasing on $(-\infty, \frac{3-\sqrt{17}}{2}]$ and on $[\frac{3+\sqrt{17}}{2}, \infty)$. The function is decreasing on $[\frac{3-\sqrt{17}}{2}, \frac{3+\sqrt{17}}{2}]$.

b) Consider the function $y(x) = x^2 - 4\ln x$. Notice that the domain is $x > 0$. The first order derivative is

$$y'(x) = 2x - \frac{4}{x}.$$

The zero of $y'(x)$ is the solution of

$$\begin{aligned} y'(x) = 0 &\Leftrightarrow 2x - \frac{4}{x} = 0 \\ &\Leftrightarrow 2x^2 - 4 = 0 \\ &\Leftrightarrow x = \sqrt{2}. \end{aligned}$$

Using the sign survey of $y'(x)$, we have the following.

$$\begin{aligned} &\text{If } x \in (0, \sqrt{2}], && y'(x) \leq 0. \\ &\text{If } x \in [\sqrt{2}, \infty), && y'(x) \geq 0. \end{aligned}$$

So, the function is increasing on $[\sqrt{2}, \infty)$ and is decreasing on $(0, \sqrt{2}]$.

c) Consider the function $y(x) = \frac{1}{2}(x^3 - x^2 + x - 2)$. The first order derivative is

$$y'(x) = \frac{1}{2}(3x^2 - 2x + 1).$$

Since the discriminant $D = 1 - 3 = -2 < 0$, $y'(x)$ has no zeros. Hence, we either have $y'(x)$ being positive or negative on the whole domain, but not both. By plugging an arbitrary x into $y'(x)$, we conclude $y'(x) > 0$ for all x. Hence, the function is increasing on $(-\infty, \infty)$.

d) Since $y(x) = xe^x$, the first order derivative is

$$y'(x) = e^x + xe^x = (1 + x)e^x.$$

The zero of $y'(x)$ is the solution of

$$\begin{aligned} y'(x) = 0 &\Leftrightarrow (1 + x)e^x = 0 \\ &\Leftrightarrow 1 + x = 0 \\ &\Leftrightarrow x = -1. \end{aligned}$$

Using the sign survey of $y'(x)$, we have the following.

$$\begin{aligned} &\text{If } x \in (-\infty, -1], && y'(x) \leq 0. \\ &\text{If } x \in [-1, \infty), && y'(x) \geq 0. \end{aligned}$$

So, the function is decreasing on $(-\infty, -1]$ and increasing on $[-1, \infty)$.

Exercise 5.2

Since $y(x) = (x - 1)(x + a)$, the first order derivative is

$$y'(x) = (x - 1) + (x + a) = 2x + a - 1.$$

In order to have the change of sign of $y'(x)$ at $x = 2$, we require

$$y'(2) = 0 \Leftrightarrow a = -3.$$

Finally, for $a = -3$, we check that indeed the following holds:

$$\begin{aligned} &y'(x) \leq 0, \text{ if } x \in (-\infty, 2]. \\ &y'(x) \geq 0, \text{ if } x \in [2, \infty). \end{aligned}$$

Hence, $a = -3$.

Exercise 5.3

Any function $y(x)$ with a *boundary* optimum in $x = b$ and $y'(b) \neq 0$ fulfills this criterion. For example, $y(x) = \sqrt{x}$. The domain is $x \geq 0$. It has a boundary minimum of 0 in $x = 0$, but $y'(0) = \frac{1}{2\sqrt{0}} \neq 0$.

Exercise 5.4

a) Consider the function $y(x) = xe^{-x^2}$. The first order derivative is

$$y'(x) = e^{-x^2} - xe^{-x^2}(2x) = e^{-x^2}(1 - 2x^2).$$

The stationary points of $y(x)$ are given by

$$y'(x) = 0 \Leftrightarrow \underbrace{e^{-x^2}}_{>0}(1-2x^2) = 0$$
$$\Leftrightarrow 1 - 2x^2 = 0$$
$$\Leftrightarrow x = \pm\frac{1}{\sqrt{2}}.$$

Using the sign survey of $y'(x)$, we have the following.

If $x \in (-\infty, -\frac{1}{\sqrt{2}}]$, $\quad y'(x) \leq 0.$

If $x \in [-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, $\quad y'(x) \geq 0.$

If $x \in [\frac{1}{\sqrt{2}}, \infty)$, $\quad y'(x) \leq 0.$

According to the monotony criterion, $y(-\frac{1}{\sqrt{2}}) = -\sqrt{\frac{1}{2e}}$ is a minimum and $y(\frac{1}{\sqrt{2}}) = \sqrt{\frac{1}{2e}}$ is a maximum.

b) Consider the function $y(x) = x^2 - 2x + 2$. The first order derivative is

$$y'(x) = 2x - 2.$$

The stationary point of $y(x)$ is in $x = 1$. Using the sign survey of $y'(x)$, we have the following:

If $x \in [0,1]$, $\quad y'(x) \leq 0.$

If $x \in [1,5]$, $\quad y'(x) \geq 0.$

According to the monotony criterion, $y(1) = 1$ is a minimum. Additionally, by use of the sign survey of $y'(x)$, we also know that $y(0) = 2$ and $y(5) = 17$ are boundary maxima.

Exercise 5.5

a) Consider the function $y(x) = \frac{1}{3}x^3 - x^2 + x + 10$. The first order derivative is

$$y'(x) = x^2 - 2x + 1 = (x-1)^2$$

Hence, the stationary point is in $x = 1$.

b) Using the sign survey of $y'(x)$, we have the following.

If $x \in [0,1]$, $\quad y'(x) \geq 0.$

If $x \in [1,\infty)$, $\quad y'(x) \geq 0.$

Since the function is increasing both on $[0,1]$ and on $[1,\infty)$, $y(x)$ has no optimum in $x = 1$. (There is only a boundary minimum $y(0) = 10$ in $x = 0$.)

Exercise 5.6

a) Consider the function $y(x) = x^2 + 2x + 2$. The first order derivative is

$$y'(x) = 2x + 2.$$

Hence, there is a stationary point in $x=-1$. Since there is only one stationary point, we can use the alternative criterion. $y(-1)=1$, $y(-2)=2$ and $y(2)=10$. Hence, $y(-1)=1$ is a minimum. Additionally, we also know that $y(-2)=2$ and $y(2)=10$ are boundary maxima.

b) Notice that the domain has no boundary points. Consider the function $y(x)=-x+\ln(x+1)$. The first order derivative is

$$y'(x)=-1+\frac{1}{x+1}.$$

The stationary point of $y(x)$ is in $x=0$. Since there is only one stationary point, we can use the alternative criterion. $y(0)=0$, $y(1)=-1+\ln 2$, $y(-\frac{1}{2})=\frac{1}{2}+\ln\frac{1}{2}$. Hence, $y(0)=0$ is a maximum.

Exercise 5.7

a) $y(x)=e^x$, $\quad y'(x)=e^x$, $\quad y''(x)=e^x$.
b) $y(x)=x\ln x$, $\quad y'(x)=\ln x+1$, $\quad y''(x)=\frac{1}{x}$.

Exercise 5.8

a) The first and second order derivatives of $y(x)$ are

$$y'(x)=3x^2-12x \quad \text{and} \quad y''(x)=6x-12.$$

The stationary points are solutions of

$$\begin{aligned} y'(x)=0 &\Leftrightarrow 3x^2-12x=0 \\ &\Leftrightarrow 3x(x-4)=0 \\ &\Leftrightarrow x=0 \text{ or } x=4. \end{aligned}$$

Since $y''(0)=-12<0$ and $y''(4)=12>0$, according to the second order criterion, $y(x)$ has a maximum $y(0)=10$ and a minimum $y(4)=-22$.

b) The first and second order derivatives of $y(x)$ are

$$\begin{aligned} y'(x)&=480x^9+56x^7-30x^5-16x^3+6x^2-4x, \\ y''(x)&=4326x^8+392x^6-150x^4-48x^2+12x-4. \end{aligned}$$

Since $y'(0)=0$ and $y''(0)=-4<0$, according to the second order criterion, $y(0)=0$ is a maximum of $y(x)$.

Exercise 5.9

Since the producer is a price taker (the per unit price is not affected by the quantity produced y), the revenue is

$$R(y)=5y.$$

Given the cost function $C(y)$, the profit equals

$$\pi(y)=5y-3y^3+2y^2.$$

According to the first order criterion the most profitable quantity produced is a stationary point of $\pi(y)$,

$$\begin{aligned} \pi'(y)=5-9y^2+4y=0 &\Leftrightarrow -9y^2+4y+5=0 \\ &\Leftrightarrow y=1 \text{ or } y=-\frac{5}{9}. \end{aligned}$$

As $-\frac{5}{9} < 0$, the only candidate is $y = 1$. Finally, we need to check whether $y = 1$ is indeed the maximum location and whether the profit is positive in $y = 1$. According to the sign survey of $\pi'(y)$ and the monotony criterion, we conclude that $\pi(1) = 4$, which is positive, is the maximum profit.

Exercise 5.10

Given the demand function $y(p) = 25 - \frac{1}{2}p$, we can express the revenue, costs, and profit functions in term of p. The revenue is given by

$$R(p) = p \cdot y(p) = 25p - \frac{1}{2}p^2,$$

the costs are

$$\begin{aligned} C(p) &= 20 + 5y(p) + \frac{1}{2}(y(p))^2 \\ &= \frac{1}{8}p^2 - 15p + 457\tfrac{1}{2}, \end{aligned}$$

and the profit equals

$$\pi(p) = R(p) - C(p) = -\frac{5}{8}p^2 + 40p - 457\tfrac{1}{2}.$$

The first order derivative of $\pi(p)$ is

$$\pi'(p) = -\tfrac{5}{4}p + 40.$$

Setting it equal to zero gives the stationary point $p = 32$. According to the sign survey of $\pi'(p)$ and the monotony criterion, we conclude that $\pi(32) = 182\frac{1}{2}$, which is positive, is the maximum profit. Thereby, the optimum prices is 32 and the optimal production level is $y(32) = 25 - \frac{32}{2} = 9$.

Exercise 5.11

a) Since the producer is a price taker, the revenue is $R(y) = py$. Given the cost function $C(y) = y\sqrt{y} = y^{\frac{3}{2}}$, the profit equals $\pi(y) = py - y^{\frac{3}{2}}$. The first order derivative is then $\pi'(y) = p - \frac{3}{2}y^{\frac{1}{2}}$. Setting it equal to zero gives $y = \frac{4}{9}p^2$. From the sign survey of $\pi'(y)$, the profit is maximal when $y = \frac{4}{9}p^2$.

   Next we determine the minimum of the average costs $AC(y) = y^{\frac{1}{2}}$. Its derivative is $AC'(y) = \frac{1}{2}y^{-\frac{1}{2}}$. Using the first order criterion, we find there is no stationary point. Using the sign survey of $AC'(y)$, the average costs has minimum $AC(0) = 0$. Hence the supply curve is defined by $y(p) = \frac{4}{9}p^2$, $p \geq 0$.

b) Since the producer is a price taker, the revenue is $R(y) = py$. Given the cost function $C(y) = 5y^3 - 40y^2 + 96y$, the profit equals $\pi(y) = py - 5y^3 + 40y^2 - 96y$. The first order derivative is then $\pi'(y) = p - 15y^2 + 80y - 96$. Setting it equal to zero gives $y = \frac{8}{3} \pm \frac{1}{30}\sqrt{640 + 60p}$. From the sign survey of $\pi'(y)$, the profit is maximal when $y = \frac{8}{3} + \frac{1}{30}\sqrt{640 + 60p}$.

   Next we determine the minimum of the average costs $AC(y) = 5y^2 - 40y + 96$. Its derivative is $AC'(y) = 10y - 40$. The first order criterion gives the stationary point.

$$AC'(y) = 0 \Leftrightarrow y = 4.$$

Using the sign survey of $AC'(y)$, the average costs has minimum $AC(4) = 16$. Hence the supply curve is defined by

$$y(p) = \begin{cases} 0 & \text{if } p < 16 \\ 2\frac{2}{3} + \frac{1}{30}\sqrt{640 + 60p} & \text{if } p \geq 16. \end{cases}$$

Exercise 5.12

$q = \sqrt{\frac{2 \cdot 5 \cdot 400}{0.1}} = 200$

Exercise 5.13

a) Because $z'_x(x,y) = 14 - 2x + y$ and $z'_y(x,y) = 2 - 2y + x$, the stationary points are solutions of

$$\begin{cases} z'_x(x,y) = 0 \\ z'_y(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 14 - 2x + y = 0 \\ 2 - 2y + x = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} 14 - 2x + y = 0 \\ x = 2y - 2 \end{cases}$$
$$\Leftrightarrow \begin{cases} 14 - 4(2y - 2) + y = 0 \\ x = 2y - 2 \end{cases}$$
$$\Leftrightarrow \begin{cases} y = 6 \\ x = 2y - 2 \end{cases}$$
$$\Leftrightarrow \begin{cases} y = 6 \\ x = 10. \end{cases}$$

Hence, $(x,y) = (10,6)$ is the only stationary point.

b) Because $z'_x(x,y) = 3x^2 - 3$ and $z'_y(x,y) = 3y^2 - 12$, the stationary points are solutions of

$$\begin{cases} z'_x(x,y) = 0 \\ z'_y(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} 3x^2 - 3 = 0 \\ 3y^2 - 12 = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} x = 1 \text{ or } x = -1 \\ y = 2 \text{ or } x = -2. \end{cases}$$

Since the partial derivatives are both independent of the other variable, all combinations of the above x's and y's are stationary points, *i.e.*, $(1,2), (1,-2), (-1,2)$ and $(-1,-2)$.

c) Because $z'_x(x,y) = y^2 + 3x^2y - y$ and $z'_y(x,y) = 2xy + x^3 - x$, the stationary points are solutions of

$$\begin{cases} z'_x(x,y) = 0 \\ z'_y(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} y^2 + 3x^2y - y = 0 \\ 2xy + x^3 - x = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} y(y + 3x^2 - 1) = 0 \\ x(2y + x^2 - 1) = 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} y = 0 \text{ or } y + 3x^2 - 1 = 0 \\ x = 0 \text{ or } 2y + x^2 - 1 = 0. \end{cases}$$

When $y = 0$, we have

$$\begin{cases} y = 0 \\ x = 0 \text{ or } 2(0) + x^2 - 1 = 0. \end{cases} \Leftrightarrow \begin{cases} y = 0 \\ x = 0 \text{ or } x = -1 \text{ or } x = 1. \end{cases}$$

When $x = 0$, we have

$$\begin{cases} y = 0 \text{ or } y + 3(0)^2 - 1 = 0 \\ x = 0 \end{cases} \Leftrightarrow \begin{cases} y = 0 \text{ or } y = 1 \\ x = 0. \end{cases}$$

When $x \neq 0$ and $y \neq 0$, we have

$$\begin{cases} y + 3x^2 - 1 = 0 \\ 2y + x^2 - 1 = 0 \end{cases} \Leftrightarrow \begin{cases} y = 1 - 3x^2 \\ 2y + x^2 - 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 1 - 3x^2 \\ 2(1 - 3x^2) + x^2 - 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 1 - 3x^2 \\ 5x^2 = \frac{1}{5} \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 1 - 3x^2 \\ x = -\frac{1}{\sqrt{5}} \text{ or } x = \frac{1}{\sqrt{5}} \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \frac{2}{5} \\ x = -\frac{1}{\sqrt{5}} \text{ or } x = \frac{1}{\sqrt{5}}. \end{cases}$$

Therefore, the stationary points are $(0, 0)$, $(-1, 0)$, $(1, 0)$, $(0, 1)$, $(-\frac{1}{\sqrt{5}}, \frac{2}{5})$ and $(\frac{1}{\sqrt{5}}, \frac{2}{5})$.

Exercise 5.14

a)

$$z(x, y) = 2xy - 5x^2 + y - 1$$
$$z'_x(x, y) = 2y - 10x$$
$$z'_y(x, y) = 2x + 1$$
$$z''_{xx}(x, y) = -10$$
$$z''_{xy}(x, y) = 2$$
$$z''_{yx}(x, y) = 2$$
$$z''_{yy}(x, y) = 0.$$

b)

$$z(x, y) = x - 5y + 3$$
$$z'_x(x, y) = 1$$
$$z'_y(x, y) = -5$$
$$z''_{xx}(x, y) = 0$$
$$z''_{xy}(x, y) = 0$$
$$z''_{yx}(x, y) = 0.$$

c)

$$
\begin{aligned}
q(L,K) &= 0.2\sqrt{LK} \\
q'_L(L,K) &= 0.2(LK)^{-\frac{1}{2}}K/2 = 0.1L^{-\frac{1}{2}}K^{\frac{1}{2}} \\
q'_K(L,K) &= 0.2(LK)^{-\frac{1}{2}}L/2 = 0.1L^{\frac{1}{2}}K^{-\frac{1}{2}} \\
q''_{LL}(L,K) &= -0.05L^{-\frac{3}{2}}K^{\frac{1}{2}} = -0.05\frac{\sqrt{K}}{L\sqrt{L}} \\
q''_{LK}(L,K) &= 0.05L^{-\frac{1}{2}}K^{-\frac{1}{2}} = 0.05\frac{1}{\sqrt{KL}} \\
q''_{KL}(L,K) &= 0.05L^{-\frac{1}{2}}K^{-\frac{1}{2}} = 0.05\frac{1}{\sqrt{KL}} \\
q''_{KK}(L,K) &= -0.05L^{\frac{1}{2}}K^{-\frac{3}{2}} = -0.05\frac{\sqrt{L}}{K\sqrt{K}}.
\end{aligned}
$$

Exercise 5.15

a) The first order partial derivatives are

$$z'_x(x,y) = 10x + 4y - 6 \quad \text{and} \quad z'_y(x,y) = 4x + 2y - 2.$$

Since $z'_x(1,-1) = z'_y(1,-1) = 0$, $(x,y) = (1,-1)$ is a stationary point of $z(x,y)$.

b) The second order partial derivatives are

$$z''_{xx}(x,y) = 10, \quad z''_{xy}(x,y) = 4 \quad \text{and} \quad z''_{yy}(x,y) = 2,$$

and hence the criterion function is given by

$$C(x,y) = 10(2) - 4^2 = 4.$$

Since $C(1,-1) = 4 > 0$ and $z''_{xx}(1,-1) = 10 > 0$, according to the second order criterion, $(1,-1)$ is a minimum location of $z(x,y)$.

Exercise 5.16

a) The second order partial derivatives are

$$z''_{xx}(x,y) = -2, \quad z''_{xy}(x,y) = 1 \quad \text{and} \quad z''_{yy}(x,y) = -2,$$

and hence the criterion function is given by

$$C(x,y) = (-2)(-2) - 1^2 = 3.$$

From Exercise 5.13, the stationary point of $z(x,y)$ is $(x,y) = (10,6)$. Since $C(10,6) = 3 > 0$ and $z''_{xx}(10,6) = -2 < 0$, $z(10,6) = 76$ is a maximum of $z(x,y)$.

b) The second order partial derivatives are

$$z''_{xx}(x,y) = 6x, \quad z''_{xy}(x,y) = 0 \quad \text{and} \quad z''_{yy}(x,y) = 6y,$$

and hence the criterion function is given by

$$C(x,y) = (6x)(6y) - 0^2 = 36xy.$$

From Exercise 5.13, the stationary points of $z(x,y)$ are $(1,2)$, $(1,-2)$, $(-1,2)$ and $(-1,-2)$. Since

$$\begin{aligned} &C(1,2) = 84 > 0, \quad z''_{xx}(1,2) = 6 > 0, \\ &C(1,-2) = -84 < 0, \\ &C(-1,2) = -84 < 0, \\ &C(-1,-2) = 84 > 0, \quad z''_{xx}(-1,-2) = -6 < 0, \end{aligned}$$

according to the second order criterion, $z(x,y)$ has a minimum $z(1,2) = 2$ and a maximum $z(-1,-2) = 38$. The points $(1,-2)$ and $(-1,2)$ are saddle points.

c) The second order partial derivatives are

$$z''_{xx}(x,y) = 6xy, \quad z''_{xy}(x,y) = 2y + 3x^2 - 1 \quad \text{and} \quad z''_{yy}(x,y) = 2x,$$

and hence the criterion function is given by

$$C(x,y) = 12x^2y - (2y + 3x^2 - 1)^2.$$

From Exercise 5.13, the stationary points of $z(x,y)$ are $(0,0)$, $(-1,0)$, $(1,0)$, $(0,1)$, $(-\frac{1}{\sqrt{5}}, \frac{2}{5})$ and $(\frac{1}{\sqrt{5}}, \frac{2}{5})$. Since

$$\begin{aligned} &C(0,0) = -1 < 0, \\ &C(0,1) = -1 < 0, \\ &C(-1,0) = -4 < 0, \\ &C(1,0) = -4 < 0, \\ &C(-\tfrac{1}{\sqrt{5}}, \tfrac{2}{5}) = \tfrac{4}{5} > 0, \quad z''_{xx}(-\tfrac{1}{\sqrt{5}}, \tfrac{2}{5}) = -2\tfrac{2}{5}\sqrt{\tfrac{1}{5}} < 0, \\ &C(\tfrac{1}{\sqrt{5}}, \tfrac{2}{5}) = \tfrac{4}{5} > 0, \quad z''_{xx}(\tfrac{1}{\sqrt{5}}, \tfrac{2}{5}) = 2\tfrac{2}{5}\sqrt{\tfrac{1}{5}} > 0, \end{aligned}$$

according to the second order criterion, $z(x,y)$ has a minimum $z(\frac{1}{\sqrt{5}}, \frac{2}{5}) = -\frac{4}{25}\sqrt{\frac{1}{5}}$ and a maximum $z(-\frac{1}{\sqrt{5}}, \frac{2}{5}) = \frac{4}{25}\sqrt{\frac{1}{5}}$. The other stationary points are saddle points.

Exercise 5.17

$\pi(L,K) = 12K^{\frac{1}{3}}L^{\frac{1}{2}} - L - \frac{2}{3}K$.
$\pi'_L(L,K) = 6K^{\frac{1}{3}}L^{-\frac{1}{2}} - 1$ and $\pi'_K(L,K) = 4K^{-\frac{2}{3}}L^{\frac{1}{2}} - \frac{2}{3}$. $\pi'_L(L,K) = 0$ gives $L = 36K^{\frac{2}{3}}$. Plugging this into $\pi'_K(L,K) = 0$ gives $K = 46656$. Then $L = 46656$ and $\pi(46656, 46656) = 15552$. To check whether this is a maximum we determine the second order partial derivatives. $\pi''_{LL}(L,K) = -3L^{-1\frac{1}{2}}K^{\frac{1}{3}}$, $\pi''_{KK}(L,K) = -2\frac{2}{3}L^{\frac{1}{2}}K^{-1\frac{2}{3}}$ and $\pi''_{LK}(L,K) = 2L^{-\frac{1}{2}}K^{-\frac{1}{3}}$. Therefore, $C(L,K) = 8L^{-1}K^{-1\frac{1}{3}} - 4L^{-1}K^{-1\frac{1}{3}} = 4L^{-1}K^{-1\frac{1}{3}} > 0$. Since, $\pi''_{LL}(L,K) < 0$, the maximum profit is 15552.

Exercise 5.18

$z(a,b) = (2 - (a \cdot 1 + b))^2 + (2 - (a \cdot 3 + b))^2 + (5 - (a \cdot 4 + b))^2 + (7 - (a \cdot 2 + b))^2$.
$z'_a(a,b) = -2(2 - a - b) - 6(2 - 3a - b) - 8(5 - 4a - b) - 4(7 - 2a - b) = -84 + 60a + 20b = 0$.
$z'_b(a,b) = -2(2 - a - b) - 2(2 - 3a - b) - 2(5 - 4a - b) - 2(7 - 2a - b) = -32 + 20a + 8b = 0$ gives $a = \frac{32}{20} - \frac{8}{20}b$. Plugging this into $z'_a(a,b) = 0$ gives $a = \frac{2}{5}$, which results in $b = 3$. Note that $z''_{aa}(a,b) = 60$, $z''_{bb}(a,b) = 8$ and $z''_{ab}(a,b) = 20$ results in $C(a,b) = 80$. Hence, $C(\frac{2}{5}, 3) = 80 > 0$ and $z''_{aa}(\frac{2}{5}, 3) = 60 > 0$, which implies that $(a,b) = (\frac{2}{5}, 3)$ is a minimum location. Consequently, $y = \frac{2}{5}x + 3$.

Exercise 5.19

a) See Chapter 7 for the line of constraint.
b) See Chapter 7 for level curves with different z-values.
c) Using the figure in Chapter 7 we obtain that $(x,y) = (5,5)$ is the minimum location with $z(5,5) = 50$.

Exercise 5.20

a) First, we solve the restriction for y,

$$y = 10 - x,$$

where $x \in [0,10]$ in order to have $x \geq 0$ and $y \geq 0$. Then, substituting the restriction into the original optimization problem $z(x,y)$ gives a optimization problem with one variable only, which is

$$\begin{aligned} &\text{optimize} \quad Z(x) = z(x, 10-x) = 2x^2 - 20x + 100, \\ &\text{where} \quad x \in [0,10]. \end{aligned}$$

The first order derivative is

$$Z'(x) = 4x - 20.$$

Hence, there is a stationary point in $x = 5$. According to the monotony criterion, it follows from the sign survey of $Z'(x)$ that the function has a minimum on $[0,10]$ at $x = 5$. The corresponding y-value is $y = 10 - 5 = 5$. And the minimum is $z(5,5) = 50$.
Additionally, by use of the sign survey of $Z'(x)$, we also know that $z(0,10) = 100$ and $z(10,0) = 100$ are boundary maxima.

b) As above, write $y = 1 - x$ and substitute into the optimization problem to give

$$\begin{aligned} &Z(x, 1-x) = -x^2 + 2x(1-x) + 2(1-x)^2 = -x^2 - 2x + 2 \\ \Rightarrow &Z'(x) = -2x - 2 \\ \Leftrightarrow &x^* = -1 \end{aligned}$$

Hence, there is a maximum $z(-1,2) = 3$.

c) Using the constraint, write $y = 12 - x$, where $x \in [0,12]$. Then write the maximization problem as

$$Z(x) = \tfrac{4}{3}x^3 - 24x^2 + 144x$$

and take the derivative as

$$Z'(x) = 4x^2 - 48x + 144x = 4(x^2 - 12x + 38) = 4(x-6)^2$$

Hence, $x = 6$ is a stationary point. However, using the sign survey shows that this is not a minimum or a maximum location. Hence, we end up with a boundary minimum $z(0,12) = 0$ and a boundary maximum $z(12,0) = 576$.

Exercise 5.21

a) Since $z'_x(x,y) = 4$, $z'_y(x,y) = 2$, $g'_x(x,y) = \frac{2}{3}x^{-\frac{1}{3}}y^{\frac{1}{3}}$ and $g'_y(x,y) = \frac{1}{3}x^{\frac{2}{3}}y^{-\frac{2}{3}}$, we have

$$\frac{z'_x(x,y)}{z'_y(x,y)} = 2 \qquad \text{and} \qquad \frac{g'_x(x,y)}{g'_y(x,y)} = \frac{2y}{x}.$$

$$\begin{cases} \frac{z'_x(x,y)}{z'_y(x,y)} = \frac{g'_x(x,y)}{g'_y(x,y)} \\ g(x,y) = k \end{cases} \Leftrightarrow \begin{cases} 2 = \frac{2y}{x} \\ x^{\frac{2}{3}}y^{\frac{1}{3}} = 10 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x \\ x^{\frac{2}{3}}y^{\frac{1}{3}} = 10 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x \\ x = 10 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 10 \\ x = 10. \end{cases}$$

After determining the possible candidate from the first order criterion, we need to check whether this candidate is a maximum, minimum or neither. Taking points to the left and the right of $x = 10$ gives the following: $z(1, 1000) = 2004$ and $z(10^{\frac{3}{2}}, 1) = 4 \cdot 10^{\frac{3}{2}} + 2$, which are both greater than $z(10, 10) = 60$. Hence, we must have $x = y = 10$ in the optimum: $z(10, 10) = 60$ is a minimum.

b) Again using the first-order criterion, we have that

$$\frac{z'_x(x,y)}{z'_y(x,y)} = \frac{2y}{2x} = \frac{g'_x(x,y)}{g'_y(x,y)} = \frac{2x}{2y} \Leftrightarrow \qquad x^2 = y^2$$

Substituting this into the constraint gives

$$2x^2 = 4 \Leftrightarrow x = \sqrt{2}$$

such that $x = y = \sqrt{2}$. Since $z(\sqrt{2}, \sqrt{2}) = 4$ and $z(2, 0) = 0$ and $z(0, 2) = 0$, we have that $z(\sqrt{2}, \sqrt{2}) = 4$ is a maximum and both $z(2, 0) = 0$ and $z(0, 2) = 0$ are boundary minima.

Exercise 5.22

The first order criterion implies

$$\frac{y}{x} = \frac{a}{b} \Leftrightarrow by = ax.$$

Plugging this into the constraint gives

$$2ax = 6.$$

Since we are looking for a and b such that the solution is in point $(9, 1)$, we plug these values into the above equation. This implies

$$18a = 6 \Leftrightarrow a = \tfrac{1}{3}.$$

Then it follows that

$$b = \frac{\frac{1}{3} \cdot 9}{1} = 3.$$

Then plugging in these values for a and b gives us the boundary values $z(0, 2) = 0$ and $z(18, 0) = 0$. Hence, we know that the maximum is indeed in $(9, 1)$ when choosing the appropriate values for a and b.

Exercise 5.23

a) $L(x,y,\lambda) = 5x^2 + 6y^2 - xy - \lambda(x + 2y - 24)$.
$L'_x(x,y,\lambda) = 10x - y - \lambda = 0$ gives $\lambda = 10x - y$. We plug this into $L'_y(x,y,\lambda) = 12y - x - 2\lambda = 0$, which gives $y = 1\frac{1}{2}x$. We plug this into $L'_\lambda(x,y,\lambda) = -x - 2y + 24 = 0$, which gives $x = 6$. Therefore, $y = 9$ and $\lambda = 51$. Hence, the stationary point of the Lagrange-function is $(x,y,\lambda) = (6,9,51)$.

b) $L(x,y,\lambda) = xy - \lambda(x^2 - 2x + y^2)$.
$L'_y(x,y,\lambda) = x - 2y\lambda = 0$ gives $\lambda = \frac{x}{2y}$. Plugging this into $L'_x(x,y,\lambda) = y - 2x\lambda + 2\lambda = 0$ gives $y^2 - x^2 + x = 0$. Putting this equal to $L'_\lambda(x,y,\lambda) = -x^2 + 2x - y^2$ results in $x(2x - 3) = 0$. Therefore, $x = 0$ or $x = 1\frac{1}{2}$. If $x = 0$, then $y = 0$ and $\lambda = 0$. If $x = 1\frac{1}{2}$, then $y = \sqrt{\frac{3}{4}}$ or $y = -\sqrt{\frac{3}{4}}$, which gives $\lambda = \sqrt{\frac{3}{4}}$ or $\lambda = -\sqrt{\frac{3}{4}}$, respectively. Hence, the stationary points of the Lagrange function are given by $(x,y,\lambda) = (0,0,0)$, $(x,y,\lambda) = (1\frac{1}{2}, \sqrt{\frac{3}{4}}, \sqrt{\frac{3}{4}})$ and $(x,y,\lambda) = (1\frac{1}{2}, -\sqrt{\frac{3}{4}}, -\sqrt{\frac{3}{4}})$.

c) $L(x,y,\lambda) = \frac{xy}{x+y} - \lambda(4x + y - 8)$.
$L'_x(x,y,\lambda) = \frac{y^2}{(x+y)^2} - 4\lambda = 0$ and $L'_y(x,y,\lambda) = \frac{x^2}{(x+y)^2} - \lambda = 0$. Combining this gives $y^2 = 4x^2$. Therefore, $y = 2x$ or $y = -2x$. Plugging $y = 2x$ into $L'_\lambda(x,y,\lambda) = -4x - y + 8 = 0$ gives $x = 1\frac{1}{3}$ in which case $y = 2\frac{2}{3}$ and $\lambda = \frac{1}{9}$. Plugging $y = -2x$ into $L'_\lambda(x,y,\lambda) = 4x + y - 8 = 0$ gives $x = 4$ in which case $y = -8$ and $\lambda = 1$. Hence, the stationary points of the Lagrange function are given by $(x,y,\lambda) = (1\frac{1}{3}, 2\frac{2}{3}, \frac{1}{9})$ and $(x,y,\lambda) = (4,-8,1)$.

Exercise 5.24

Solving the first order criterion of the constrained optimization problem gives

$$\begin{cases} \frac{U'_x(x,y)}{U'_y(x,y)} = \frac{p_x}{p_y} \\ p_x x + p_y y = I \end{cases} \Leftrightarrow \begin{cases} \frac{y^2}{2xy} = \frac{4}{2} \\ 4x + 2y = 48 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 4x \\ 2x + y = 24 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 4x \\ 6x = 24 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 4x \\ x = 4. \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 16 \\ x = 4. \end{cases}$$

Hence the point $(x,y) = (4,16)$ is a candidate for the maximum location. The other candidates are from the boundary: $(0,24)$ and $(12,0)$. Since $U(4,16) = 1024 > U(0,24) = U(12,0) = 0$, the function $U(x,y)$ has a maximum $U(4,16) = 1024$.

Exercise 5.25

Solving the first order criterion of the constrained optimization problem gives

$$\begin{cases} \frac{U'_r(r,b)}{U'_b(r,b)} = \frac{p_r}{p_b} \\ p_r r + p_b b = I \end{cases} \Leftrightarrow \begin{cases} \frac{2r}{4b} = \frac{10}{20} \\ 10r + 20b = 90 \end{cases}$$

$$\Leftrightarrow \begin{cases} r = b \\ r + 2b = 9 \end{cases}$$

$$\Leftrightarrow \begin{cases} r = b \\ 3b = 9 \end{cases}$$

$$\Leftrightarrow \begin{cases} r = 3 \\ b = 3. \end{cases}$$

Hence the point $(3,3)$ is a candidate for the maximum location. The other candidates are from the boundary: $(0, 4\frac{1}{2})$ and $(9,0)$. Since $U(9,0) = 81 > U(0, 4\frac{1}{2}) = 40\frac{1}{2} > U(3,3) = 27$, the function $U(x,y)$ has a maximum $U(9,0) = 81$.

Exercise 5.26

The optimization problem in this exercise is

minimize $\quad C(L,K) = 3L + 12K$
subject to $\quad X(L,K) = 10,$
where $L, K > 0.$

Solving the first order criterion of the constrained optimization problem together with the equality constraint gives

$$\begin{cases} \frac{X'_L(L,K)}{X'_K(L,K)} = \frac{w}{r} \\ X(L,K) = 10 \end{cases} \Leftrightarrow \begin{cases} \frac{L^{-\frac{1}{2}}K^{\frac{1}{2}}/2}{L^{\frac{1}{2}}K^{-\frac{1}{2}}/2} = \frac{3}{12} \\ L^{\frac{1}{2}}K^{\frac{1}{2}} = 10 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{K}{L} = \frac{1}{4} \\ L^{\frac{1}{2}}K^{\frac{1}{2}} = 10 \end{cases}$$

$$\Leftrightarrow \begin{cases} L = 4K \\ 2K = 10 \end{cases}$$

$$\Leftrightarrow \begin{cases} L = 20 \\ K = 5. \end{cases}$$

Hence the point $(20,5)$ is a candidate for the minimum of the optimization problem, with $C(20,5) = 120$. To show $(20,5)$ is indeed a minimum location, we take values on either side of $L = 20$. For example, $C(10,10) = 150$ and $C(100,1) = 300$. Hence, $(20,5)$ is a minimum location and the producer has to pay at least 120 to produce 10 units.

Exercise 5.27

a) Solving the first order criterion of the constrained optimization problem together with the equality constraint gives

$$\begin{cases} \frac{P'_L(L,K)}{P'_K(L,K)} = \frac{w}{r} \\ P(L,K) = y \end{cases} \Leftrightarrow \begin{cases} \frac{L^{-\frac{1}{2}}/2}{K^{-\frac{1}{2}}/2} = \frac{2}{1} \\ L^{\frac{1}{2}} + K^{\frac{1}{2}} = y \end{cases}$$

$$\Leftrightarrow \begin{cases} K^{\frac{1}{2}} = 2L^{\frac{1}{2}} \\ L^{\frac{1}{2}} + K^{\frac{1}{2}} = y \end{cases}$$

$$\Leftrightarrow \begin{cases} K^{\frac{1}{2}} = 2L^{\frac{1}{2}} \\ 3L^{\frac{1}{2}} = y \end{cases}$$

$$\Leftrightarrow \begin{cases} K^{\frac{1}{2}} = 2L^{\frac{1}{2}} \\ L = \frac{1}{9}y^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} K = \frac{4}{9}y^2 \\ L = \frac{1}{9}y^2. \end{cases}$$

If the costs are minimal at $(L,K) = (\frac{1}{9}y^2, \frac{4}{9}y^2)$, then the cost function $C(y)$ is given by

$$C(y) = C(\tfrac{1}{9}y^2, \tfrac{4}{9}y^2) = \tfrac{2}{3}y^2.$$

We check whether this is the minimum cost function by taking values both to the left and the right of $L = \frac{1}{9}y^2$: $C(y^2, 0) = 2y^2$ and $C(0, y^2) = y^2$. Hence, $(L,K) = (\frac{1}{9}y^2, \frac{4}{9}y^2)$ is the minimum location and $C(y) = \frac{2}{3}y^2$ is the cost function.

b) The profit equals $\pi(y) = py - C(y) = py - \frac{2}{3}y^2$. The first order derivative is then $\pi'(y) = p - \frac{4}{3}y$. Setting it equal to zero gives $y = \frac{3}{4}p$. From the sign survey of $\pi'(y)$, the profit is maximal when $y = \frac{3}{4}p$.
The average costs are $AC(y) = \frac{C(y)}{y} = \frac{\frac{2}{3}y^2}{y} = \frac{2}{3}y$. From $AC'(y) = \frac{2}{3} > 0$, we have $AC'(y) > 0$ for all $y \geq 0$. Hence, the minimum of the average costs is at the boundary point $y = 0$ with value $AC(0) = 0$. Hence the supply curve is given by $y(p) = \frac{3}{4}p$, $(p \geq 0)$.

Exercise 5.28
$U(\mu,\sigma) = \mu - 5\sigma^2 = 0.08w_1 + 0{,}1w_2 - 5 \cdot 0.05^2 \cdot w_2^2 = 0.08 + 0.02w_2 - 0.0125w_2^2 = u(w_2)$.
$u_2'(w_2) = 0.02 - 0.025w_2 = 0$ gives $w_2 = 0.8$ and $w_1 = 1 - 0.8 = 0.2$.
Since $u''(w_2) = -0.25 < 0$ this is a maximum location.

Exercise 5.29
$U(\mu,\sigma) = \mu - \frac{1}{2}\alpha\sigma^2 = \mu_S + w_2(\mu_A - \mu_S) - \frac{1}{2}\alpha \cdot w_2^2\sigma_A^2 = u(w_2)$.
$u_2'(w_2) = \mu_A - \mu_S - \alpha w_2 \sigma_A^2 = 0$ (and $u_2''(w_2) = -\alpha\sigma_A^2 < 0$) gives $w_2 = \dfrac{\mu_A - \mu_S}{\alpha\sigma_A^2} > 0$.

Exercise 5.30

$$\begin{aligned} U(\mu,\sigma) &= \mu - 5\sigma^2 \\ &= 0.04w_1 + 0.06w_2 + 0.08w_3 - 10\Big(\sqrt{\tfrac{9}{2500}w_2^2 + \tfrac{4}{625}w_3^2 - \tfrac{1}{500}w_2w_3}\Big)^2 \\ &= 0.04 + 0.02w_2 + 0.04w_3 - \tfrac{9}{250}w_2^2 - \tfrac{40}{625}w_3^2 + \tfrac{1}{50}w_2w_3 \\ &= u(w_2, w_3). \end{aligned}$$

$u'_2(w_2, w_3) = 0.02 - 0.072w_2 + 0.02w_3 = 0$ gives $w_3 = 3.6w_2 - 1$. We plug this into $u'_3(w_2, w_3) = 0.04 - 0.128w_3 + 0.02w_2 = 0$ to obtain $w_2 = \frac{210}{551}$, which gives $w_3 = \frac{205}{551}$ and $w_1 = 1 - w_2 - w_3 = \frac{136}{551}$.

We still have to check whether this is a maximum location: $u''_{22}(w_2, w_3) = -0.072$, $u''_{33}(w_2, w_3) = -0.128$ and $u''_{23} = 0.02$ gives $C(w_2, w_3) = (-0.072)(-0.128) - 0.02^2 = 0.008816 > 0$. Since $u''_{22}(w_2, w_3) < 0$ we find that $w_1 = \frac{136}{551}$, $w_2 = \frac{210}{551}$, $w_3 = \frac{205}{551}$ indeed gives a maximum.

Exercise 5.31

a) $y'(x) = e^{-x}$, $y''(x) = -e^{-x} < 0$ for all x. Hence, $y(x)$ is concave.
b) $y'(x) = 6x^5 + 4x^3$, $y''(x) = 30x^4 + 16x^2 \geq 0$ for all x. Hence, $y(x)$ is convex.
c) $y'(x) = -2x + 4$, $y''(x) = -2 < 0$ for all x. Hence, $y(x)$ is concave.
d) $y'(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $y''(x) = -\frac{1}{4}x^{-1\frac{1}{2}} < 0$ for all $x > 0$. For $x = 0$ the function $y''(x)$ is not defined. However, $y(x) = \sqrt{x}$ is a standard function of which we know the graph and for every two points on this graph the line piece connecting the two points is below the graph of the function. Hence, $y(x)$ is concave.

Exercise 5.32

a) Since $y(x) = xe^{-\frac{1}{2}x^2}$, we have

$$y'(x) = e^{-\frac{1}{2}x^2} + xe^{-\frac{1}{2}x^2}(-x) = e^{-\frac{1}{2}x^2}(1 - x^2),$$
$$y''(x) = -2xe^{-\frac{1}{2}x^2} + (x^3 - x)e^{-\frac{1}{2}x^2} = e^{-\frac{1}{2}x^2}(x^3 - 3x).$$

The zeros of the second order derivative are given by

$$\begin{aligned} y''(x) = 0 &\Leftrightarrow x^3 - 3x = 0 \\ &\Leftrightarrow x(x^2 - 3) = 0 \\ &\Leftrightarrow x = 0 \text{ or } x = \sqrt{3} \text{ or } x = -\sqrt{3}. \end{aligned}$$

According to the second order criterion of a convex / concave function and the sign survey of $y''(x)$, $y(x)$ is concave on $(-\infty, -\sqrt{3}]$ and $[0, \sqrt{3}]$ and convex on $[-\sqrt{3}, 0]$ and $[\sqrt{3}, \infty)$.

b) Since $y(x) = (\ln x)^2$, we have

$$y'(x) = \frac{2}{x}\ln x,$$
$$y''(x) = \frac{2}{x} \cdot \frac{1}{x} + 2(\ln x) \cdot \frac{-1}{x^2} = \frac{2}{x^2}(1 - \ln x).$$

The zeros of the second order derivative are given by

$$\begin{aligned} y''(x) = 0 &\Leftrightarrow \frac{2}{x^2}(1 - \ln x) = 0 \\ &\Leftrightarrow \ln x = 1 \\ &\Leftrightarrow x = e. \end{aligned}$$

According to the second order criterion of a convex / concave function and the sign survey of $y''(x)$, $y(x)$ is on convex on $(0, e]$ and concave on $[e, \infty)$.

Exercise 5.33

a) $y'(x) = e^{-x}$ is never equal to zero. Hence, $y(x)$ does not have an extremum.

b) $y'(x) = 6x^5 + 4x^3 = 0$ gives $x^3(6x^2+4) = 0$ and therefore, $x = 0$ or $6x^2 + 4 = 0$. Since $6x^2 + 4 > 0$ for all x, $x = 0$ is the only stationary point. Since the function is convex $x = 0$ is a minimum location.

c) $y'(x) = -2x + 4 = 0$ gives $x = 2$. Since the function is concave $x = 2$ is a maximum location.

d) $y'(x) = \frac{1}{2}x^{-\frac{1}{2}} > 0$ for all $x > 0$. For $x = 0$ the function $y'(x)$ is not defined. However, since $x \geq 0$ is the domain of $y(x)$ and $y'(x) = \frac{1}{2}x^{-\frac{1}{2}} > 0$ for all $x > 0$ it holds that $y(x) = \sqrt{x}$ has a boundary minimum at $x = 0$.

Exercise 5.34

a) The first and second order partial derivatives are given by

$$z'_x(x,y) = 2x + 2y + 4, \qquad z'_y(x,y) = 2x + 4y - 2,$$
$$z''_{xx}(x,y) = 2, \qquad z'_{xy}(x,y) = 2, \qquad z'_{yy}(x,y) = 4.$$

Since

$$C(x,y) = 2(4) - 2^2 = 4 > 0, \quad z''_{xx}(x,y) > 0 \text{ and } z''_{yy} > 0,$$

according to the second order criterion, the function $z(x,y)$ is convex.

b) The first and second order partial derivatives are given by

$$z'_x(x,y) = -6x + 2y, \qquad z'_y(x,y) = 2x - 4y,$$
$$z''_{xx}(x,y) = -6, \qquad z''_{xy}(x,y) = 2, \qquad z''_{yy}(x,y) = -4.$$

Since

$$C(x,y) = (-4)(-6) - 2^2 = 20 > 0, \quad z''_{xx}(x,y) < 0 \text{ and } z''_{yy} < 0,$$

according to the second order criterion, the function $z(x,y)$ is concave.

c) The first and second order partial derivatives are given by

$$z'_x(x,y) = 2x + 4y, \qquad z'_y(x,y) = 4x - 2y,$$
$$z''_{xx}(x,y) = 2, \qquad z''_{xy}(x,y) = 4, \qquad z''_{yy}(x,y) = -2.$$

Since $C(x,y) = -2 \cdot 2 - 4^2 = -20 < 0$ the function $z(x,y)$ is neither convex nor concave.

d) The first and second order partial derivatives are given by

$$z'_x(x,y) = 2x - \frac{2}{x^2y}, \qquad z'_y(x,y) = 2y - \frac{2}{xy^2},$$
$$z''_{xx}(x,y) = 2 + \frac{4}{x^3y}, \qquad z''_{xy}(x,y) = \frac{2}{x^2y^2}, \qquad z''_{yy}(x,y) = 2 + \frac{4}{xy^3}.$$

Since

$$C(x,y) = \left(2 + \frac{4}{x^3y}\right)\left(2 + \frac{4}{xy^3}\right) - \frac{4}{x^4y^4} = 4 + \frac{8}{x^3y} + \frac{8}{xy^3} + \frac{12}{x^4y^4} > 0,$$
$$z''_{xx}(x,y) > 0 \qquad z''_{yy}(x,y) > 0$$

for $x, y > 0$, according to the second order criterion, the function $z(x,y)$ is convex.

Exercise 5.35

The first and second order partial derivatives are given by

$$z'_x(x,y) = 2px + 2py, \qquad z'_y(x,y) = 2px + 2y,$$
$$z''_{xx}(x,y) = 2p, \qquad z''_{xy}(x,y) = 2p, \qquad z''_{yy}(x,y) = 2.$$

If the function is convex, according to the second order criterion, we have

$$\begin{cases} C(x,y) \geq 0 \\ z''_{xx}(x,y) \geq 0 \\ z''_{yy}(x,y) \geq 0 \end{cases} \Leftrightarrow \begin{cases} 2p(2) - (2p)^2 \geq 0 \\ 2p \geq 0 \\ 2 > 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} 4p(1-p) \geq 0 \\ p \geq 0 \end{cases}$$
$$\Leftrightarrow \begin{cases} 0 \leq p \leq 1 \\ p \geq 0 \end{cases}$$
$$\Leftrightarrow 0 \leq p \leq 1.$$

Exercise 5.36

a) The condition for stationary points is

$$2x + 2y + 4 = 0 \Leftrightarrow 2x = -2y - 4$$
$$2x + 4y - 2 = 0$$

Combining these two equations gives

$$2y - 4 + 4y - 2 = 0 \Leftrightarrow y = 3, x = -5$$

The stationary point as $(x,y) = (-5,3)$. Since the function is convex (see Exercise 5.34), this is a minimum location.

b) The condition for stationary points is

$$-6x + 2y = 0$$
$$2x - 4y = 0$$

Hence, the stationary point is $(x,y) = (0,0)$, which is a maximum location due to concavity.

c) The condition for stationary points is

$$2x + 4y = 0$$
$$4x - 2y = 0$$

There is a saddle point at $(x,y) = (0,0)$.

d) The appropriate condition is

$$2x - \frac{2}{x^2y} = 0 \Leftrightarrow 2x = \frac{2}{x^2y} \Leftrightarrow y = \frac{1}{x^3}$$
$$2y - \frac{2}{xy^2} = 0 \Leftrightarrow 2y = \frac{2}{xy^2} \Leftrightarrow x^8 = 1$$
$$\Rightarrow y = 1 \text{ or } y = -1 \qquad x = 1 \text{ or } x = -1$$

But recall that $x, y > 0$. Hence the only stationary point is $(x,y) = (1,1)$, which is a minimum location due to convexity.

Exercise 5.37

Consider the function $y(x) = e^{-\frac{1}{3}(x^3+ax-7)}$. The first order derivative is

$$y'(x) = e^{-\frac{1}{3}(x^3+ax-7)} \cdot (-\tfrac{1}{3}(3x^2+a)).$$

To have $y(x)$ decreasing on the whole domain, we need $y'(x) \leq 0$ for all x. It implies

$$\begin{aligned} y'(x) \leq 0 &\Leftrightarrow \underbrace{e^{-\frac{1}{3}(x^3+ax-7)}}_{>0} - \tfrac{1}{3}(3x^2+a) \leq 0 \\ &\Leftrightarrow \underbrace{-\tfrac{1}{3}}_{<0}(3x^2+a) \leq 0 \\ &\Leftrightarrow 3x^2+a \geq 0 \\ &\Leftrightarrow a \geq -3x^2 \end{aligned}$$

for all x. Since $-3x^2$ has a maximum 0 in $x=0$, we have

$$\begin{aligned} a \geq -3x^2 &\Leftrightarrow a \geq 0 \geq -3x^2 \\ &\Leftrightarrow a \geq 0. \end{aligned}$$

Exercise 5.38

If $p = 0$, then $y'(x) = 4x$, which implies that $y(x)$ is decreasing on $(-\infty, 0]$. If $p \neq 0$, then $y'(x) = 3px^2 + 4x = 0$ gives $x = 0$ or $x = -\frac{4}{3p}$. If $p > 0$, then $y'(x) < 0$ for $-\frac{4}{3p} < x < 0$. If $p < 0$, then $y'(x) < 0$ for $x < 0$. Hence, such a p does not exist.

Exercise 5.39

Consider the function $y(x) = \ln(\frac{1}{x}) + \frac{x^2}{8}$, $(x > 0)$. The first order derivative is

$$y'(x) = \frac{1}{\frac{1}{x}}(-\frac{1}{x^2}) + \tfrac{2}{8}x = -\frac{1}{x} + \tfrac{1}{4}x.$$

The stationary points of $y(x)$ are given by

$$\begin{aligned} y'(x) = 0 &\Leftrightarrow -\frac{1}{x} + \tfrac{1}{4}x = 0 \\ &\Leftrightarrow -4 + x^2 = 0 \\ &\Leftrightarrow x = 2. \end{aligned}$$

Using the sign survey of $y'(x)$, we have the following.

If $x \in (0, 2]$,	$y'(x) \leq 0$.
If $x \in [2, \infty)$,	$y'(x) \geq 0$.

According to the monotony criterion, $y(2) = \ln(\frac{1}{2}) + \frac{1}{2}$ is a minimum.

Exercise 5.40

$y'(x) = 6x \cdot 5^{3x^2} \ln 5$. Hence, $y'(x) = 0$ if $x = 0$. It follows that $y'(x) < 0$ for $x < 0$ and $y'(x) > 0$ for $x > 0$. Hence, $y(0) = 1$ is a minimum. Further, $y(-1) = 125$ is a boundary maximum.

Exercise 5.41

The first and second order derivatives are

$$y'(x) = -8x^7 + 7x^6 + 6x^5 + 15x^4 - 16x^3 - 6x^2 + 2x,$$
$$y''(x) = -56x^6 + 42x^5 + 30x^4 - 16x^3 - 48x^2 - 12x + 2.$$

Since

$$y'(1) = -8 + 7 + 6 + 15 - 16 - 6 + 2 = 0,$$
$$y''(1) = -56 + 42 + 30 + 60 - 48 - 12 + 2 = 18 > 0,$$

according to the second order criterion, $y(1)$ is a minimum of $y(x)$.

Exercise 5.42

a) $\pi(y) = 25y - 5y^2 + 7y - 10$. $\pi'(y) = -10y + 32 = 0$ gives $y = 3\frac{1}{5}$. $\pi''(y) = -10 < 0$. Hence, $\pi(3\frac{1}{5}) = 41\frac{1}{5}$ is the maximum profit.

b) $p = \frac{100}{y}$. Therefore, $\pi(y) = py - C(y) = 100 - 5y^2 + 7y - 10$. $\pi'(y) = -10y + 7 = 0$ gives $y = \frac{7}{10}$. $\pi''(y) = -10 < 0$. Hence, the maximum profit is $\pi(\frac{7}{10}) = 92\frac{9}{20}$.

Exercise 5.43

a) $AC(y) = \frac{1}{3}y^2 - 3y + 9$. $AC'(y) = \frac{2}{3}y - 3 = 0$ gives $y = 4\frac{1}{2}$. $AC''(y) = \frac{2}{3} > 0$. Hence, $AC(4\frac{1}{2}) = 2\frac{1}{4}$ is the minimum price for which the entrepreneur makes a profit. Hence, he will not make a profit for $p \leq 2\frac{1}{4}$.

b) $\pi(y) = py - C(y) = -\frac{1}{3}y^3 + 3y^2 - (p-9)y$. $\pi'(y) = -y^2 + 6y + p - 9$. $D = 6^2 - 4(-1)(p-9) = 4p$. Therefore, $y = \frac{-6 \pm \sqrt{4p}}{-2}$. Since $\pi'(y)$ is a quadratic function with a negative coefficient for the y^2-term, we know that $y = \frac{-6+\sqrt{4p}}{-2}$ is a minimum location and $y = \frac{-6-\sqrt{4p}}{-2} = 3 + \sqrt{p}$ a maximum location. This gives
$$y(p) = \begin{cases} 0 & \text{if } p < 2\frac{1}{4} \\ 3 + \sqrt{p} & \text{if } p \geq 2\frac{1}{4}. \end{cases}$$

Exercise 5.44

a) Consider the cost function $C(y) = \frac{1}{10}y^3 - 3y^2$. To check whether there will be production, we compute the minimum of the average costs $AC(y) = \frac{1}{10}y^2 - 3y + 50$. The derivative of $AC(y)$ is $AC'(y) = \frac{1}{5}y - 3$. The first order criterion gives the stationary point.

$$AC'(y) = 0 \Leftrightarrow y = 15.$$

Using the sign survey of $AC'(y)$, the average costs has a minimum $AC(15) = 27\frac{1}{2}$. Since the market price $p = 30 > 27\frac{1}{2}$, there will indeed be production.

b) Profit is equal to zero when $p = AC$, *i.e.*, $p = 27\frac{1}{2}$.

c) Since the producer is a price taker, the revenue is $R(y) = py$. Given the cost function $C(y) = \frac{1}{10}y^3 - 3y^2 + 50y$, the profit equals $\pi(y) = py - \frac{1}{10}y^3 + 3y^2 - 50y$. The first order derivative is then $\pi'(y) = p - \frac{3}{10}y^2 + 6y - 50$. Setting it equal to zero gives $y = 10 \pm \frac{5}{3}\sqrt{\frac{6}{5}p - 24}$. From the sign survey of $\pi'(y)$, the profit is maximal when $y = 10 + \frac{5}{3}\sqrt{\frac{6}{5}p - 24}$.

From part a), the average costs has minimum $27\frac{1}{2}$. Hence the supply function is defined by

$$y(p)=\begin{cases}0 & \text{if } p<27\frac{1}{2}\\ 10+2\frac{2}{3}\sqrt{1\frac{1}{5}p-24} & \text{if } p\geq 27\frac{1}{2}.\end{cases}$$

Exercise 5.45

a) $q_A=\sqrt{\frac{2c_Ad_A}{h_A}}=\sqrt{\frac{2\cdot 500\cdot 1000}{1}}=1000$. Hence, shop A orders once a year.

b) $q_B=1000=\sqrt{\frac{2\cdot 500 d_B}{1.5}}$ gives $1000000=\frac{1000d_B}{1.5}$, and hence, $d_B=1500$. Therefore, the fixed yearly demand is 1500.

Exercise 5.46

a) The stationary points of $z(x,y)$ are solutions of

$$\begin{cases}z'_x(x,y)=0\\ z'_y(x,y)=0\end{cases} \Leftrightarrow \begin{cases}3x^2-3y^4=0\\ -12xy^3-384=0\end{cases}$$
$$\Leftrightarrow \begin{cases}x^2=y^4\\ xy^3=-32\end{cases}$$
$$\Leftrightarrow \begin{cases}x=-y^2 \text{ or } x=y^2\\ xy^3=-32.\end{cases}$$

When $x=-y^2$, we have

$$\begin{cases}x=-y^2\\ xy^3=-32\end{cases} \Leftrightarrow \begin{cases}x=-y^2\\ -y^5=-32\end{cases}$$
$$\Leftrightarrow \begin{cases}x=-y^2\\ y=2\end{cases}$$
$$\Leftrightarrow \begin{cases}x=-4\\ y=2.\end{cases}$$

When $x=y^2$, we have

$$\begin{cases}x=y^2\\ xy^3=-32\end{cases} \Leftrightarrow \begin{cases}x=y^2\\ y^5=-32\end{cases}$$
$$\Leftrightarrow \begin{cases}x=y^2\\ y=-2\end{cases}$$
$$\Leftrightarrow \begin{cases}x=4\\ y=-2.\end{cases}$$

Hence, the stationary point are $(-4,2)$ and $(4,-2)$.

b) The second order partial derivatives are

$$z''_{xx}(x,y)=6x,\quad z''_{xy}(x,y)=-12y^3 \quad\text{and}\quad z''_{yy}(x,y)=-36xy^2,$$

and hence the criterion function is given by

$$C(x,y)=-216x^2y^2-144y^6.$$

Since $C(-4,2) < 0$ and $C(4,-2) < 0$, both stationary points are not extrema but saddle points. Therefore, the function has no optima.

Exercise 5.47

$z'_y(x,y) = 2x + 2y = 0$ gives $y = -x$. Plugging this into $z'_x(x,y) = 3x^2 - 4x + 2y = 0$ gives $3x^2 - 4x - 2x = 0$. Hence, $x = 0$ or $x = 2$. The stationary points are therefore $(x,y) = (0,0)$ and $(x,y) = (2,-2)$.

$z''_{xx}(x,y) = 6x - 4$, $z''_{yy} = 2$, $z''_{xy}(x,y) = 2$. Therefore, $C(x,y) = 12(x-1)$. $C(0,0) = -12 < 0$. Hence, $(x,y) = (0,0)$ is a saddle point. $C(2,-2) = 12 > 0$, $z''_{xx}(2,-2) = 8 > 0$ and hence, $z(2,-2) = -6$ is a minimum.

Exercise 5.48

The first order partial derivatives are

$$z'_x(x,y) = 2px + 2py \quad \text{and} \quad z'_y(x,y) = 2px + 2y.$$

Since $z'_x(0,0) = z'_y(0,0) = 0$, the point $(0,0)$ is a stationary point. The second order partial derivatives are

$$z''_{xx}(x,y) = 2p, \quad z''_{xy}(x,y) = 2p \quad \text{and} \quad z''_{yy}(x,y) = 2,$$

and hence the criterion function is

$$C(x,y) = 2(2p) - (2p)^2 = 4p - 4p^2 = 4p(1-p).$$

For a minimum, we require $z''_{xx}(x,y) > 0$ and $C(x,y) > 0$. Both conditions are fulfilled for $0 < p < 1$. For $C(x,y) = 0$ we can not use the second order criterion. $C(x,y) = 0$ if $p = 0$ or if $p = 1$. If $p = 0$, then $z(x,y) = y^2$ and $z(0,0) = 0$ is clearly a minimum. If $p = 1$, then $z(x,y) = (x+y)^2$. Also in that case $z(0,0) = 0$ is clearly a minimum. Hence, $z(0,0)$ is a minimum for $0 \le p \le 1$.

Exercise 5.49

$\pi(L,K) = 16L^{\frac{1}{4}}K^{\frac{1}{4}} - 2L - 4K$. Hence, $\pi'_L(L,K) = 4L^{-\frac{3}{4}}K^{\frac{1}{4}} - 2$ and $\pi'_K(L,K) = 4L^{\frac{1}{4}}K^{-\frac{3}{4}} - 4$. Then $\pi'_K(L,K) = 4L^{\frac{1}{4}}K^{-\frac{3}{4}} - 4 = 0$ results in $L = K^3$. Plugging this into $\pi'_L(L,K) = 4L^{-\frac{3}{4}}K^{\frac{1}{4}} - 2 = 0$ results in $K = \sqrt{2}$ and hence, $L = 2\sqrt{2}$. Via the second order partial derivatives we obtain the criterium function $C(L,K) = 8L^{-\frac{3}{2}}K^{-\frac{3}{2}} > 0$. Moreover, $\pi''_{LL}(L,K) = -3L^{-\frac{7}{4}}K^{\frac{1}{4}} < 0$, which implies that $(L,K) = (2\sqrt{2},\sqrt{2})$ is a maximum location. Then the maximum profit is $\pi(2\sqrt{2},\sqrt{2}) = 8\sqrt{2}$.

Exercise 5.50

We first find the minimum cost function: $\dfrac{\frac{1}{2}L^{-\frac{1}{2}}K^{\frac{1}{4}}}{\frac{1}{4}L^{\frac{1}{2}}K^{-\frac{3}{4}}} = \frac{5}{10}$ gives $4K = L$. Substituting in the production function gives $y = (4K)^{\frac{1}{2}}K^{\frac{1}{4}} = 2K^{\frac{3}{4}}$. Rewriting gives $K = 2^{-\frac{4}{3}}y^{\frac{4}{3}}$. Hence, $L = 4K = 2^{\frac{2}{3}}y^{\frac{4}{3}}$.

This gives cost function $C(y) = C(2^{\frac{2}{3}}y^{\frac{4}{3}}, 2^{-\frac{4}{3}}y^{\frac{4}{3}}) = 5 \cdot 2^{\frac{2}{3}}y^{\frac{4}{3}} + 10 \cdot 2^{-\frac{4}{3}}y^{\frac{4}{3}} = 30 \cdot 2^{-\frac{4}{3}}y^{\frac{4}{3}}$. Since $C(y^{\frac{4}{3}}, y^{\frac{4}{3}}) = 25y^{\frac{4}{3}}$ and $C(4y^{\frac{4}{3}}, \frac{1}{16}y^{\frac{4}{3}}) = 20\frac{5}{8}y^{\frac{4}{3}}$ result in higher cost functions, $C(y) = 30 \cdot 2^{-\frac{4}{3}}y^{\frac{4}{3}}$ is indeed the minimum cost function.

The marginal cost function is then given by

$$MC(y) = C'(y) = \tfrac{4}{3} \cdot 30 \cdot 2^{-\frac{4}{3}} y^{\frac{1}{3}} = 40 \cdot 2^{-\frac{4}{3}} y^{\frac{1}{3}} = 10 \cdot (4y)^{\frac{1}{3}}$$

while average costs are

$$AC(y) = 30 \cdot 2^{-\frac{4}{3}} y^{\frac{1}{3}}$$

Hence, $AC'(y)$ is never zero for $y > 0$ and there is a minimum in $y = 0$. The supply function therefore follows from $p = MC(y)$. Rewriting $p = 10 \cdot (4y)^{\frac{1}{3}}$ gives $y(p) = \frac{p^3}{4000}$, $(p \geq 0)$.

Exercise 5.51

$z(a,b) = (4-(a\cdot 1+b))^2 + (3-(a\cdot 2+b))^2 + (4-(a\cdot 3+b))^2 + (1-(a\cdot 4+b))^2$.
$z'_a(a,b) = -2(4-a-b) - 4(3-2a-b) - 8(1-4a-b) = -52+60a+20b = 0$ gives $a = \frac{52}{60} - \frac{1}{3}b$. We plug this into $z'_b(a,b) = -2(4-a-b) - 2(3-2a-b) - 2(4-3a-b) - 2(1-4a-b) = -24+20a+8b = 0$, which gives $b = 5$ and $a = -\frac{4}{5}$. Of course, $z''_{aa}(a,b) = 60$, $z''_{bb}(a,b) = 8$ and $z''_{ab}(a,b) = 20$ gives $C(a,b) = 80 > 0$. Combining this with $z''_{aa}(a,b) = 60 > 0$ gives that we have found a minimum location. Hence, $y = -\frac{4}{5}x + 5$.

Exercise 5.52

$z(a,b) = (0-(a\cdot 1+b))^2 + (-1-(a\cdot 2+b))^2 + (3-(a\cdot 3+b))^2 + (q_4-(a\cdot 4+b))^2$.
$z'_a(a,b) = -2(0-a-b) - 4(-1-2a-b) - 6(3-3a-b) - 8(q_4-4a-b) = -14-8q_4+60a+20b = 0$. Since $a = 1$ and $b = -1\frac{1}{2}$ this gives $q_4 = 2$.

Exercise 5.53

Solving the first order criterion of the constrained optimization problem together with the equality constraint gives

$$\begin{aligned}
\begin{cases} \frac{z'_x(x,y)}{z'_y(x,y)} = \frac{g'_x(x,y)}{g'_y(x,y)} \\ g(x,y) = k \end{cases}
&\Leftrightarrow \begin{cases} \frac{4x(4x^2+y^2)}{y(4x^2+y^2)} = 2 \\ 2x+y = 12 \end{cases} \\
&\Leftrightarrow \begin{cases} \frac{4x}{y} = 2 \\ 2x+y = 12 \end{cases} \\
&\Leftrightarrow \begin{cases} y = 2x \\ 2x+y = 12 \end{cases} \\
&\Leftrightarrow \begin{cases} y = 2x \\ 4x = 12 \end{cases} \\
&\Leftrightarrow \begin{cases} y = 6 \\ x = 3. \end{cases}
\end{aligned}$$

Hence the point $(3,6)$ is a candidate for the optimum location. The other possible candidates are from the boundary: $(0,12)$ and $(6,0)$. Since $U(3,6) = 6\sqrt{2} < U(6,0) = U(0,12) = 12$, the function $U(x,y)$ has a minimum $U(3,6) = 6\sqrt{2}$ and boundary maxima $U(6,0) = U(0,12) = 12$.

Exercise 5.54

$\dfrac{1}{3\cdot\frac{1}{2\sqrt{y}}} = \dfrac{4}{3}$ gives $\frac{2}{3}\sqrt{y} = \frac{4}{3}$ and hence, $y = 4$. Via the restriction we then obtain $x = 3$. Since $z(0,8) = 3\sqrt{8}$ and $z(6,0) = 6$, the maximum is $z(3,4) = 9$.

Exercise 5.55
$\frac{1}{2} = \frac{2x}{6y}$ gives $x = 1\frac{1}{2}y$. We plug this into the restriction to obtain $y = 2$. Hence, $x = 3$. Since $z(2,3) = 8$ and $z(0,\sqrt{7}) = 2\sqrt{7}$, the minimum is $z(\sqrt{21},0) = \sqrt{21}$.

Exercise 5.56
$L(x,y,\lambda) = x^2 + xy^2 - \lambda(x+y-4)$.
$L'_x(x,y,\lambda) = 2x + y^2 - \lambda = 0$ and $L'_y(x,y,\lambda) = 2xy - \lambda = 0$. Hence, $2x + y^2 = 2xy$, which gives $x = \frac{y^2}{2(y-1)}$. We plug this into $L'_\lambda(x,y,\lambda) = x + y - 4 = 0$, which gives $3y^2 - 10y + 8 = 0$. Using the 'abc'-formula we obtain $y = 2$ or $y = 1\frac{1}{3}$. From $y = 2$ it follows that $x = 2$ and $\lambda = 8$. From $y = 1\frac{1}{3}$ it follows that $x = 2\frac{2}{3}$ and $\lambda = 7\frac{1}{9}$. Hence, the stationary points of the Lagrange function are $(x,y,\lambda) = (2,2,8)$ and $(x,y,\lambda) = (2\frac{2}{3},1\frac{1}{3},7\frac{1}{9})$.

Exercise 5.57
Due to the government regulation, the constrained optimization problem becomes

$$\begin{array}{ll} \text{maximize} & U(x,y) = x^2y^3 \\ \text{subject to} & x + 3y = 10 \\ \text{where} & x \geq 0, 0 \leq y \leq 1. \end{array}$$

Solving the first order criterion together with the equality constraint gives

$$\begin{cases} \frac{U'_x(x,y)}{U'_y(x,y)} = \frac{p_x}{p_y} \\ p_x x + p_y y = 10 \end{cases} \Leftrightarrow \begin{cases} \frac{2xy^3}{3x^2y^2} = \frac{1}{3} \\ x + 3y = 10 \end{cases}$$
$$\Leftrightarrow \begin{cases} \frac{2y}{3x} = \frac{1}{3} \\ x + 3y = 10 \end{cases}$$
$$\Leftrightarrow \begin{cases} x = 2y \\ x + 3y = 10 \end{cases}$$
$$\Leftrightarrow \begin{cases} x = 2y \\ 5y = 10 \end{cases}$$
$$\Leftrightarrow \begin{cases} x = 4 \\ y = 2. \end{cases}$$

However, y should not exceed one. Hence, the point $(4,2)$ is *not* a candidate for the maximum location. The other candidates are the boundary points: $(7,1)$ and $(10,0)$. Note that $(0,\frac{10}{3})$ is not considered here, because $\frac{10}{3} > 1$. Since $U(7,1) = 49 > U(10,0) = 0$, the function $U(x,y)$ has a maximum $U(7,1) = 49$.

Exercise 5.58
$\dfrac{2x}{2y} = \dfrac{2(x-1)}{2(y-1)}$ gives $y = 2x$. We plug this into the restriction, which gives $(x-1)^2 + (2x-2)^2 = 10$. This results in $x^2 - 2x - 1 = 0$. The solutions are $x = 1 \pm \sqrt{2}$. Since $x \geq 0$ only $x = 1 + \sqrt{2}$ is possible and $y = 2 + 2\sqrt{2}$ follows from that value of x. Then since $z(0,5) = 25$ and $z(\sqrt{6}+1,0) = 7 + 2\sqrt{6}$, the maximum is $z(1+\sqrt{2}, 2+2\sqrt{2}) = 15 + 10\sqrt{2}$.

Exercise 5.59

a) Solving the first order criterion of the constrained optimization problem together with the equality constraint gives

$$\begin{cases} \frac{U'_x(x,y)}{U'_y(x,y)} = \frac{p_x}{p_y} \\ p_x x + p_y y = I \end{cases} \Leftrightarrow \begin{cases} \frac{y}{x} = 1 \\ x + y = 12 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x \\ x + y = 12 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = x \\ 2y = 12 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 6 \\ x = 6. \end{cases}$$

Hence the point $(6, 6)$ is a candidate for the maximum location. The other possible candidates are from the boundary: $(0, 12)$ and $(12, 0)$. Since $U(6, 6) = 36 > U(12, 0) = U(0, 12) = 0$, the utility function $U(x, y)$ has a maximum at $(x_0, y_0) = (6, 6)$ with utility level 36.

b) Solving the first order criterion of the constrained optimization problem together with the equality constraint gives

$$\begin{cases} \frac{U'_x(x,y)}{U'_y(x,y)} = \frac{p_x}{p_y} \\ p_x x + p_y y = I \end{cases} \Leftrightarrow \begin{cases} \frac{y}{x} = 3 \\ 3x + y = 12 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 3x \\ 3x + y = 12 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 3x \\ 6x = 12 \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 6 \\ x = 2. \end{cases}$$

Hence the point $(2, 6)$ is a candidate for the maximum location. The other possible candidates are from the boundary: $(0, 12)$ and $(4, 0)$. Since $U(2, 6) = 12 > U(4, 0) = U(0, 12) = 0$, the utility function $U(x, y)$ has a maximum at $(x_2, y_2) = (2, 6)$ with utility level 12.

c) Solving the first order criterion of the constrained optimization problem together with the equality constraint gives

$$\begin{cases} \frac{U'_x(x,y)}{U'_y(x,y)} = \frac{p_x}{p_y} \\ p_x x + p_y y = I \end{cases} \Leftrightarrow \begin{cases} \frac{y}{x} = 3 \\ 3x + y = I \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 3x \\ 3x + y = I \end{cases}$$

$$\Leftrightarrow \begin{cases} y = 3x \\ 6x = I \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \frac{I}{2} \\ x = \frac{I}{6}. \end{cases}$$

Hence the point $(\frac{I}{6}, \frac{I}{2})$ is a candidate for the maximum location. The other possible candidates are from the boundary: $(0, I)$ and $(\frac{I}{3}, 0)$. Since $U(2,6) = \frac{1}{12}I^2 \geq U(\frac{I}{3}, 0) = U(0, I) = 0$, the utility function $U(x, y)$ has a maximum at $(\frac{I}{6}, \frac{I}{2})$ with utility level $\frac{1}{12}I^2$.
To solve for the income level that would allow for the initial level of capital, we now need

$$\frac{1}{12}I'^2 = 36 \qquad \Rightarrow I' = 12\sqrt{3}$$

The required transfer is then

$$I' - 12 = 12\sqrt{3} - 12.$$

Furthermore, $x_1 = \frac{12\sqrt{3}}{6} = 2\sqrt{3}$ and $y_1 = \frac{12\sqrt{3}}{2} = 6\sqrt{3}$.

d) The substitution effect of good x is $x_1 - x_0 = 2\sqrt{3} - 6$.
The substitution effect of good y is $y_1 - y_0 = 6\sqrt{3} - 6$.
The income effect of good x is $x_2 - x_1 = 2 - 2\sqrt{3}$.
The income effect of good y is $y_2 - y_1 = 6 - 6\sqrt{3}$.
The total effect of good x is $x_2 - x_0 = 2 - 6 = -4$.
The total effect of good y is $y_2 - y_0 = 6 - 6 = 0$.

Exercise 5.60

Solving the first order criterion of the constrained optimization problem together with the equality constraint gives

$$\begin{cases} \frac{U'_x(x,y)}{U'_y(x,y)} = \frac{p_x}{p_y} \\ p_x x + p_y y = I \end{cases} \Leftrightarrow \begin{cases} \frac{\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{4}}}{\frac{1}{4}x^{\frac{1}{2}}y^{-\frac{3}{4}}} = \frac{4}{3} \\ 4x + 3y = I \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{2y}{x} = \frac{4}{3} \\ 4x + 3y = I \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \frac{2x}{3} \\ 4x + 3y = I \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \frac{2x}{3} \\ 6x = I \end{cases}$$

$$\Leftrightarrow \begin{cases} y = \frac{I}{9} \\ x = \frac{I}{6}. \end{cases}$$

The point $(\frac{I}{6}, \frac{I}{9})$ is a candidate for the maximum location. The other candidates are from boundary: $(0, \frac{I}{3})$ and $(\frac{I}{4}, 0)$. Since $U(\frac{I}{6}, \frac{I}{9}) = \frac{I^{\frac{3}{4}}}{6^{\frac{1}{2}}9^{\frac{1}{4}}} \geq U(0, \frac{I}{3}) = U(\frac{I}{4}, 0) = 0$ for $I \geq 0$, the function $U(x, y)$ has a maximum at $(\frac{I}{6}, \frac{I}{9})$. Hence, $U(\frac{I}{6}, \frac{I}{9}) = (\frac{1}{6})^{\frac{1}{2}}(\frac{1}{9})^{\frac{1}{4}}I^{\frac{3}{4}}$.

Exercise 5.61

a) $\dfrac{K(K+2L) - LK \cdot 2}{L(K+2L) - LK} = \dfrac{K^2}{2L^2} = \dfrac{1}{8}$ gives $8K^2 = 2L^2$ and hence, $L = 2K$. Plugging this into the production function gives $K = 2\frac{1}{2}Y$. Hence, $L = 5Y$. Hence, $C(Y) = (5 + 8 \cdot 2\frac{1}{2})Y = 25Y$. This is the minimum cost function as $C(\frac{5}{3}Y, 5Y) = 41\frac{2}{3}Y$ and $C(10Y, 2\frac{1}{4}Y) = 28Y$.

b) $p(Y) = 31 - \frac{1}{2}Y$. Hence, $\pi(Y) = pY - C(Y) = 31Y - \frac{1}{2}Y^2 - 25Y$, which gives $\pi'(Y) = -y + 6 = 0$. Since $\pi''(Y) = -1 < 0$ it holds that $\pi(6) = 18$ is the maximum profit.

Exercise 5.62

a) $U(\mu_S, \sigma_S) = U(0.05, 0) = 0.05 < 0.08 = 0.1 - 2 \cdot 0.1^2 = U(0.1, 0.1) = U(\mu_A, \sigma_A)$. Hence, the investor prefers investing in stock A.

b) $U(\mu, \sigma) = 0.05w_1 + 0.1w_2 - 2 \cdot 0.1^2 w_2^2 = 0.05 + 0.05w_2 - 0.02w_2^2 = u(w_2)$. Hence, $u'_{w_2}(w_2) = 0.05 - 0.04w_2 = 0$ gives $w_2 = 1\frac{1}{4}$. Hence, there is no stationary point in the domain. Hence, the maximum is located at the boundary. By part a) we know the investor prefers stock A to savings account S, hence $w_1 = 0$, $w_2 = 1$.

Exercise 5.63

a) $U(\mu_S, \sigma_S) = U(0.05, 0) = 0.05$, $U(\mu_{A_1}, \sigma_{A_1}) = U(0.1, 0.1) = 0.1 - 4 \cdot 0.1^2 = 0.06$ and $U(\mu_{A_2}, \sigma_{A_2}) = U(0.05, 0.1) = 0.1 - 4 \cdot 0.1^2 = 0.02$. Hence, the investor prefers investing in stock A_1.

b) $U(w_1, w_2, w_3) = 0.05w_1 + 0.1w_2 + 0.05w_3 - 0.04w_2^2 - 0.04w_3^2 + 0.02w_2w_3 = 0.05 + 0.05w_2 - 0.04w_2^2 - 0.04w_3^2 + 0.02w_2w_3 = u(w_2, w_3)$. From $u'_3(w_2, w_3) = -0.08w_3 + 0.02w_2 = 0$ we obtain $w_2 = 4w_3$. Plugging this into $u'_2(w_2, w_3) = 0.05 - 0.08w_2 + 0.02w_3 = 0$ gives $w_3 = \frac{1}{6}$ and hence, $w_2 = \frac{2}{3}$ and $w_1 = \frac{1}{6}$.

Further, $u''_{22}(w_2, w_3) = -0.08$, $u''_{33}(w_2, w_3) = -0.08$ and $u''_{23}(w_2, w_3) = 0.02$ gives $C(w_2, w_3) = 0.006 > 0$, which in combination with $u''_{22}(w_2, w_3) < 0$ gives that $(w_2, w_3) = (\frac{2}{3}, \frac{1}{6})$ is a minimum location of $u(w_2, w_3)$. Hence, $w_1 = \frac{1}{6}$, $w_2 = \frac{2}{3}$, $w_3 = \frac{1}{6}$.

Exercise 5.64

$y'(x) = -4x + 3 + \frac{1}{x}$, $y''(x) = -4 - \frac{1}{x^2} < 0$ for all x in the domain. Hence, $y(x)$ is never convex, $y(x)$ is concave if $x > 0$.

Exercise 5.65

$$y'(x) = 2x - 10 + \frac{8}{x}$$
$$y''(x) = 2 - \frac{8}{x^2}$$

If $x > 2$, then $x^2 > 4$ and $\frac{8}{x^2} < 2$, hence $y''(x) > 0$, implying convexity.

Exercise 5.66

Via $z'_x(x, y) = 3x^2 + 3y$ and $z'_y(x, y) = 3x + 6y^2$ we get $z''_{xx}(x, y) = 6x$, $z''_{yy}(x, y) = 12y$, $z''_{xy}(x, y) = 3$. Therefore, $C(x, y) = 72xy - 9$. $C(x, y) \geq 0$ if $xy \geq \frac{1}{8}$. Then the second order partial derivatives determine whether the function is convex or concave. Hence, $z(x, y)$ is convex if $y \geq \frac{1}{8x}$ and $x > 0$, $z(x, y)$ is concave if $y \leq \frac{1}{8x}$ and $x < 0$.

Exercise 5.67

$$z'_x(x,y) = 2ax + 6y \qquad z'_y(x,y) = 6x + 2y$$
$$z''_{xx}(x,y) = 2a \qquad z''_{xy}(x,y) = 6 \qquad z''_{yy}(x,y) = 2$$
$$\Rightarrow C(x,y) = 4a - 36 \geq 0 \Leftrightarrow a \geq 9$$
$$z''_{xx}(x,y) = 2a \geq 0 \Leftrightarrow a \geq 0$$
$$z''_{x}(x,y) = 2 > 0 \text{ for all } a$$

Hence, the function is convex for $a \geq 9$.

Exercise 5.68

a)

$$z'_x(x,y) = 2 - 2x + 2y \qquad z'_y(x,y) = -20 + 2x - 8y$$
$$z''_{xx}(x,y) = -2 < 0 \qquad z''_{xy}(x,y) = 2 \qquad z''_{yy}(x,y) = -8 < 0$$
$$C(x,y) = 28 > 0$$

Hence, the function is concave.

b) We need

$$2 - 2x + 2y = 0$$

and

$$-20 + 2x - 8y = 0$$

to hold simultaneously. Solving this system of equations, for example by adding both equations up, gives the stationary point as

$$y = -3 \qquad x = -2$$

This is a maximum, due to the concavity of the function. Hence, $z(-2,-3) = 28$ is a maximum value. There is no minimum value.

6

Solutions to Chapter 6

Exercise 6.1

a) $F(x) = 5x + c$, where c is an arbitrary constant.

b) $F(x) = 1\frac{1}{2}x^2 + c$, where c is an arbitrary constant.

c) $F(x) = 2x^3 + c$, where c is an arbitrary constant.

Exercise 6.2

a) $\int_{-2}^{6} 5\ dx = [5x]_{-2}^{6} = 5(6) - 5(-2) = 40.$

b) $\int_0^4 3x\ dx = \left[1\frac{1}{2}x^2\right]_0^4 = 1\frac{1}{2}(4)^2 - 1\frac{1}{2}(0)^2 = 24.$

c) $\int_1^3 6x^2\ dx = \left[2x^3\right]_1^3 = 2(3^3) - 2(1)^3 = 52.$

Exercise 6.3

a) $f(x) = x\sqrt{x} = x^{\frac{3}{2}}, \quad F(x) = \left(\frac{2}{5}\right) x^{\frac{5}{2}} = \frac{2}{5}x^2\sqrt{x}.$

b) $f(x) = e^{-\frac{1}{2}x}, \quad F(x) = -2e^{-\frac{1}{2}x}.$

c) $f(x) = \frac{1}{x^2} = x^{-2}, \quad F(x) = -x^{-1} = -\frac{1}{x}.$

d) $q(p) = \sqrt[3]{p^2} = p^{\frac{2}{3}}, \quad Q(p) = \frac{3}{5}p^{\frac{5}{3}} = \frac{3}{5}p\sqrt[3]{p^2}.$

e) $w(L) = L^3\sqrt{L} = L^{\frac{7}{2}}, \quad W(L) = \frac{2}{9}L^{\frac{9}{2}} = \frac{2}{9}L^4\sqrt{L}.$

f) $y(x) = 4^{-x} = \left(\frac{1}{4}\right)^x, \quad Y(x) = \frac{4^{-x}}{\ln\frac{1}{4}} = -\frac{4^{-x}}{\ln 4}.$

Exercise 6.4

a) $\int_0^1 x\sqrt{x}\ dx = \left[\frac{2}{5}x^2\sqrt{x}\right]_0^1 = \frac{2}{5}.$

b) $\int_1^2 e^{-\frac{1}{2}x}\ dx = \left[-2e^{-\frac{1}{2}x}\right]_1^2 = (-2e^{-\frac{1}{2}\cdot 2}) - (-2e^{-\frac{1}{2}\cdot 1}) = 2(e^{-\frac{1}{2}} - e^{-1}).$

c) $\int_1^4 \frac{1}{x^2}\ dx = \left[-\frac{1}{x}\right]_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}.$

d) $\int_0^8 \sqrt[3]{p^2}\ dp = \left[\frac{3}{5}p^{\frac{5}{3}}\right]_0^8 = \frac{96}{5} = 19\frac{1}{5}.$

e) $\int_0^{64} L\sqrt{L}\ dL = \left[\frac{2}{9}L^4\sqrt{L}\right]_0^{64} = \frac{2^{28}}{9} = 29826161\frac{7}{9}.$

f) $\int_{-1}^1 4^{-x}\ dx = \left[-\frac{4^{-x}}{\ln 4}\right]_{-1}^1 = (-\frac{4^{-1}}{\ln 4}) - (-\frac{4^1}{\ln 4}) = \frac{3\frac{3}{4}}{\ln 4}.$

Exercise 6.5

a) $G(t) = 1\frac{1}{3}e^{3t} + 3\ln t.$

b) $F(x) = 1\frac{1}{2}x^2 + \frac{2^x}{\ln 2}.$

c) $y(t) = \left(\sqrt{t} + \frac{1}{t}\right)^2 = t + \frac{2}{\sqrt{t}} + \frac{1}{t^2}$, $Y(t) = \frac{1}{2}t^2 + 4\sqrt{t} - \frac{1}{t}.$

Exercise 6.6

a) $\int_1^e 4e^{3t} + \frac{3}{t}\ dt = \left[1\frac{1}{3}e^{3t} + 3\ln t\right]_1^e = 1\frac{1}{3}e^{3e} + 3 - 1\frac{1}{3}e^3 = 1\frac{1}{3}(e^{3e} - e^3) + 3.$

b) $\int_1^2 3x + 2^x\ dx = \left[1\frac{1}{2}x^2 + \frac{2^x}{\ln 2}\right]_1^2 = 4\frac{1}{2} + \frac{2}{\ln 2}.$

c) $\int_1^4 \left(\sqrt{t} + \frac{1}{t}\right)^2\ dx = \left[\frac{1}{2}t^2 + 4\sqrt{t} - \frac{1}{t}\right]_1^4 = 12\frac{1}{4}.$

Exercise 6.7

a) $O(f,0,3) = \int_0^3 5\ dx = [5x]_0^3 = 15.$

b) $O(f,0,16) = \int_0^{16} x\ dx = [\frac{1}{2}x^2]_0^{16} = \frac{1}{2}16^2 = 128.$

c) $O(f,0,1) = \int_0^1 3x^2\ dx = [x^3]_0^1 = 1.$

Exercise 6.8

a) $O(f,0,4) = -\int_0^4 -2\ dx = -[-2x]_0^4 = 8.$

b) $O(f,1,4) = -\int_1^4 -x\ dx = -\left[-\frac{1}{2}x^2\right]_1^4 = 7\frac{1}{2}.$

Exercise 6.9

a) The zeros of the function $f(x)$ are $x = 0$, $x = 1$ and $x = 3$. From the sign survey of $f(x)$, the function is positive on $[0,1]$, negative on $[1,3]$ and positive on $[3,4]$.The area is therefore given by

$$\begin{aligned} O(f,0,4) &= \int_0^1 x^3 - 4x^2 + 3x\ dx - \int_1^3 x^3 - 4x^2 + 3x\ dx + \int_3^4 x^3 - 4x^2 + 3x\ dx \\ &= \left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2\right]_0^1 - \left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2\right]_1^3 + \left[\tfrac{1}{4}x^4 - \tfrac{4}{3}x^3 + \tfrac{3}{2}x^2\right]_3^4 \\ &= \tfrac{5}{12} - \left(-2\tfrac{1}{4} - \tfrac{5}{12}\right) + \left(2\tfrac{2}{3} - \left(-2\tfrac{1}{4}\right)\right) \\ &= 8. \end{aligned}$$

.

b) The zeros of the function $f(x)$ are $x = 1$ and $x = 4$. From the sign survey of $f(x)$, the function is positive on $[0,1]$, negative on $[1,4]$ and positive on $[4,6]$. The area is therefore given by

$$\begin{aligned} O(f,0,6) &= \int_0^1 x^2 - 5x + 4\ dx - \int_1^4 x^2 - 5x + 4\ dx + \int_4^6 x^2 - 5x + 4\ dx \\ &= \left[\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 4x\right]_0^1 - \left[\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 4x\right]_1^4 + \left[\tfrac{1}{3}x^3 - \tfrac{5}{2}x^2 + 4x\right]_4^6 \\ &= 15. \end{aligned}$$

c) The zero of the function $f(x)$ is $x = 1$. From the sign survey of $f(x)$, the function is negative on $[0,1]$ and positive on $[1,4]$. The area is therefore given by

$$\begin{aligned} O(f,0,4) &= -\int_0^1 e^x - e\ dx + \int_1^4 e^x - e\ dx \\ &= -[e^x - ex]_0^1 + [e^x - ex]_1^4 \\ &= -(e - e - 1) + e^4 - 4e - 0 \\ &= e^4 - 4e + 1. \end{aligned}$$

Exercise 6.10

a) First, we determine all possible intersection points. Since

$$\begin{aligned} f(x) = g(x) &\Leftrightarrow 2x - 1 = -2x + 2 \\ &\Leftrightarrow x = \frac{3}{4}, \end{aligned}$$

the intersection point of $f(x)$ and $g(x)$ is $(\frac{3}{4}, \frac{1}{2})$. Similarly, $(\frac{1}{2}, 0)$ and $(1, 0)$ are the intersection points of $f(x)$ and the x-axis, and of $g(x)$ and the x-axis, respectively.
Therefore, the area is given by

$$\begin{aligned} \text{Area I} + \text{Area II} &= \int_{\frac{1}{2}}^{\frac{3}{4}} f(x) - 0\, dx + \int_{\frac{3}{4}}^{1} g(x) - 0\, dx \\ &= \int_{\frac{1}{2}}^{\frac{3}{4}} 2x - 1\, dx + \int_{\frac{3}{4}}^{1} -2x + 2\, dx \\ &= \left[x^2 - x\right]_{\frac{1}{2}}^{\frac{3}{4}} + \left[-x^2 + 2x\right]_{\frac{3}{4}}^{1} \\ &= \tfrac{9}{16} - \tfrac{3}{4} - \tfrac{1}{4} + \tfrac{1}{2} - 1 + 2 + \tfrac{9}{16} - \tfrac{3}{2} \\ &= \tfrac{1}{8}. \end{aligned}$$

b) We proceed in the similar manner as above. The intersection points of $f(x)$ and $g(x)$ are given by

$$\begin{aligned} f(x) = g(x) &\Leftrightarrow x^2 = x \\ &\Leftrightarrow x = 0 \text{ or } x = 1. \end{aligned}$$

From the sign survey of $f(x) - g(x)$, the enclosed area is equal to

$$\begin{aligned} \int_0^1 g(x) - f(x)\, dx &= \int_0^1 x - x^2\, dx \\ &= \left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 \\ &= \tfrac{1}{2} - \tfrac{1}{3} \\ &= \tfrac{1}{6}. \end{aligned}$$

Exercise 6.11

We replace the upper bound with t,

$$\int_1^t \frac{1}{x^3}\, dx = \int_1^t x^{-3}\, dx = \left[-\tfrac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2t^2} + \tfrac{1}{2}.$$

As $t \to \infty$, we have $2t^2 \to \infty$ and hence, $-\frac{1}{2t^2} \to 0$. Consequently,

$$\int_1^\infty \frac{1}{x^3}\, dx = 0 + \tfrac{1}{2} = \tfrac{1}{2}.$$

Exercise 6.12

a) The consumer surplus is the surface area enclosed by the graph of the inverse demand function, the p-axis and the line $p = 25$. See Chapter 7 for the figure.

b) Solving the demand function for variable p gives

$$\begin{aligned} q = 36 - 4\sqrt{p} &\Leftrightarrow 4\sqrt{p} = 36 - q \\ &\Leftrightarrow \sqrt{p} = 9 - \frac{q}{4} \\ &\Leftrightarrow p = \left(9 - \tfrac{1}{4}q\right)^2. \end{aligned}$$

Hence, the inverse demand function is $p(q) = \left(9 - \frac{1}{4}q\right)^2$ with $0 \leq q \leq 36$. Given the equilibrium price $p = 25$, the equilibrium quantity is $q = 36 - 4\sqrt{25} = 16$. We calculate the consumer surplus as the integral of the inverse demand function $p(q)$ minus the function $p = 25$ over the interval $[0, 16]$, which is

$$\int_0^{16} \left(9 - \tfrac{1}{4}q\right)^2 \, dq = \int_0^{16} 81 - \tfrac{9}{2}q + \tfrac{1}{16}q^2 - 25 \, dq = \left[81q - \tfrac{9}{4}q^2 + \tfrac{1}{48}q^3\right]_0^{16} = 405\tfrac{1}{3}.$$

c) Now we calculate the consumer surplus as the integral of the demand function $q(p)$ over the interval $[25, 81]$, which is

$$\int_{25}^{81} 36 - 45\sqrt{p} \, dp = \left[36p - \frac{45}{\frac{3}{2}}p^{\frac{3}{2}}\right]_{25}^{81} = 972 - 566\tfrac{2}{3} = 405\tfrac{1}{3}.$$

Exercise 6.13

First of all, $f(x) \geq 0$ for all $x \geq 0$.
Further, $\int_0^t 7e^{-7x} \, dx = \left[-7e^{-7x}\right]_0^t = -e^{-7t} + e^0 = -e^{-7t} + 1$. If $t \to \infty$, then $-7e^{-7t} \to 0$, which means we can conclude that $\int_0^\infty 7e^{-7x} \, dx = 1$.

Exercise 6.14

$\int_2^5 c \, dx = [cx]_2^5 = 3c = 1$, hence $c = \frac{1}{3}$. Note that $f(x) = \frac{1}{3} \geq 0$ on the whole domain.

Exercise 6.15

$P[Y < 5] = P[Y \leq 5] = \int_0^5 7e^{-7x} \, dx = \left[-7e^{-7x}\right]_0^5 = (-e^{-35}) - (-1) = 1 - e^{-35}$.

Exercise 6.16

$\int_0^4 f(x) \, dx = [F(x)]_0^4 = \ln 17 - \ln 1 = \ln 17$.

Exercise 6.17

The statement is false, since

$$(F(x)G(x))' = F(x)g(x) + f(x)G(x)$$

which is not equal to $f(x)g(x)$ in general. To verify, take for example $f(x) = g(x) = 1$.

Exercise 6.18

The zeros of the function $f(x) = x(x-3)(x-4)$ are $x = 0$, $x = 3$ and $x = 4$. From the sign survey of $f(x)$, the function is positive on $[0, 3]$ and $[4, \infty)$ and negative on $(-\infty, 0]$ and $[3, 4]$. The enclosed area is equal to $\int_0^p f(x)$ if $f(x) \geq 0$ on $[0, p]$, which implies $0 \leq p \leq 3$.

Exercise 6.19

Since $f(x) = e^{x^3+x^2+5x}$, $f(0) = 1$ and $f(1) = e^7$. Due to the convexity of $f(x)$, the curve $f(x)$ lies *below* the line segment connecting $(0,1)$ and $(1,e^7)$ in the interval $[0,1]$. This line is given by

$$y = (e^7 - 1)x + 1.$$

Therefore, we have

$$\begin{aligned}\int_0^1 f(x)\ dx &< \int_0^1 ((e^7-1)x+1)\ dx \\ &= \left[\tfrac{1}{2}(e^7-1)x^2 + x\right]_0^1 \\ &= \tfrac{1}{2}(e^7-1)+1 \\ &= \tfrac{1}{2}(e^7+1).\end{aligned}$$

Exercise 6.20

The enclosed area is

$$\begin{aligned}\int_1^4 f(x) - g(x)\ dx &= \int_1^4 \sqrt{x} - \frac{1}{x^2}\ dx \\ &= \left[\tfrac{2}{3}x\sqrt{x} + \frac{1}{x}\right]_1^4 \\ &= \tfrac{16}{3} + \tfrac{1}{4} - \tfrac{2}{3} - 1 \\ &= 3\tfrac{11}{12}.\end{aligned}$$

Exercise 6.21

a) $\int_0^2 x^2 - 3x + 2\ dx = \left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 2x\right]_0^2 = \tfrac{8}{3} - 6 + 4 = \tfrac{2}{3}$.

b) The zeros of the function $f(x) = x^2 - 3x + 2$ are $x = 1$ and $x = 2$. From the sign survey of $f(x)$, the function is positive on $[0,1]$ and negative on $[1,2]$. The area is therefore given by

$$\begin{aligned}O(y,0,2) &= \int_0^1 (x^2-3x+2)\ dx - \int_1^2 (x^2-3x+2)\ dx \\ &= \left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 2x\right]_0^1 - \left[\tfrac{1}{3}x^3 - \tfrac{3}{2}x^2 + 2x\right]_1^2 \\ &= \tfrac{1}{3} - \tfrac{3}{2} + 2 - \tfrac{8}{3} + 6 - 4 + \tfrac{1}{3} - \tfrac{3}{2} + 2 \\ &= 1.\end{aligned}$$

Exercise 6.22

$$\begin{aligned}f(x) = g(x) &\Leftrightarrow -2x + 9 = x^2 + 1 \\ &\Leftrightarrow x^2 + 2x - 8 = 0 \\ &\Leftrightarrow (x+4)(x-2) = 0.\end{aligned}$$

Hence, $x = -4$ and $x = 2$. Then the area is given by

$-\int_{-4}^2 x^2 + 2x - 8\ dx = -\left[\tfrac{1}{3}x^3 + x^2 - 8x\right]_{-4}^2 = -((-9\tfrac{1}{3}) - (26\tfrac{2}{3})) = 36.$

Exercise 6.23

$\int_1^4 (\frac{1}{\sqrt{x}} + \frac{1}{2x})\ dx = \left[2x^{\frac{1}{2}} + \frac{1}{2}\ln(x)\right]_1^4 = 2 + \frac{1}{2}\ln(4)$.

Exercise 6.24

$$\begin{aligned} f(x) = g(x) &\Leftrightarrow x^3 = x \\ &\Leftrightarrow x^3 - x = 0 \\ &\Leftrightarrow x(x^2-1) = 0. \end{aligned}$$

Hence, $x = 0$, $x = -1$ and $x = 1$. Then Area I is given by $\int_{-1}^0 x^3 - x\ dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-1}^0 = (0-0) - (\frac{1}{4} - \frac{1}{2}) = \frac{1}{4}$. Further, Area II is given by $\int_0^1 x - x^3\ dx = \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = (\frac{1}{2} - \frac{1}{4}) - (0-0) = \frac{1}{4}$. Hence, the total area is $\frac{1}{2}$.

Exercise 6.25

We replace the upper bound with t,

$$\int_0^t e^{-x}\ dx = \left[-e^{-x}\right]_0^t = -\frac{1}{e^t} + 1.$$

As $t \to \infty$, we have $e^t \to \infty$ and hence, $-\frac{1}{e^t} \to 0$. Therefore,

$$\int_0^\infty e^{-x}\ dx = 0 + 1 = 1.$$

Exercise 6.26

$\int_t^{-3} (e^x - \frac{5}{x^7})\ dx = \left[e^x + \frac{5}{6}x^{-6}\right]_t^{-3} = (e^{-3} + \frac{5}{6}(-3)^{-6}) - (e^t - \frac{5}{6}t^{-6})$. As $t \to -\infty$, then $e^t \to 0$ and $\frac{5}{6}t^{-6} \to 0$. Hence, the area is given by $e^{-3} + \frac{5}{6}(-3)^{-6} = \frac{5}{4374} + \frac{1}{e^3}$.

Exercise 6.27

$\int_1^t f(x)\ dx = \left[\frac{5+x}{2x}\right]_1^t = \frac{5+t}{2t} - \frac{5+1}{2\cdot 1} = \frac{5}{2t} + \frac{t}{2t} - 3 = -2\frac{1}{2} + \frac{5}{2t}$. If $t \to \infty$, then $\frac{5}{2t} \to 0$. Hence, $\int_1^\infty f(x)\ dx = -2\frac{1}{2}$.

Exercise 6.28

Demand and supply are in equilibrium when

$$\begin{aligned} X_d(p) = X_s(p) &\Leftrightarrow 12 - 3\sqrt{p} = \sqrt{p} \\ &\Leftrightarrow 4\sqrt{p} = 12 \\ &\Leftrightarrow \sqrt{p} = 3 \\ &\Leftrightarrow p = 9. \end{aligned}$$

Hence, the equilibrium price is $p = 9$. Since the zero of the demand function $X_d(p)$ is $p = 16$, we can calculate the consumer surplus as the integral of the demand function $X_d(p)$ over the interval $[9, 16]$, which is

$$\begin{aligned}\int_9^{16} X_d(p)\ dp =& \int_9^{16} (12 - 3\sqrt{p})\ dp \\ =& \left[12p - 2p^{\frac{3}{2}}\right]_9^{16} \\ =& (12 \cdot 16 - 2 \cdot 16^{\frac{3}{2}}) - (12 \cdot 9 - 2 \cdot 9^{\frac{3}{2}}) \\ =& 10.\end{aligned}$$

Exercise 6.29

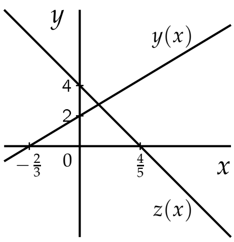
$\int_0^{10} cx\ dx = \left[\frac{1}{2}cx^2\right]_0^{10} = 50c = 1$. Hence, $c = \frac{1}{50}$, in which case $f(x) \geq 0$ for all $0 \leq x \leq 10$. Hence, $c = \frac{1}{50}$.

Exercise 6.30

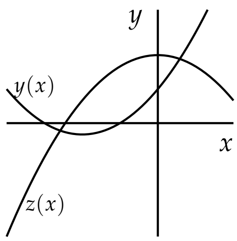
$\int_2^{t} \frac{c}{x^4}\ dx = \left[-\frac{c}{3}x^{-3}\right]_2^{t} = (-\frac{c}{3}t^{-3}) - (-\frac{c}{3} \cdot \frac{1}{8}) = 1$ gives $c = 24$. Then $f(x) \geq 0$ for all $x \geq 0$. Hence, $c = 24$.

7 Figures

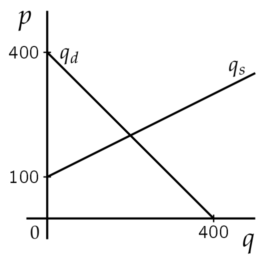
1.1a)



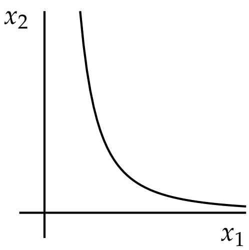
1.4a)



1.19c)



3.2a)



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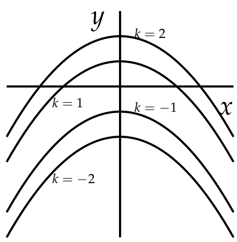

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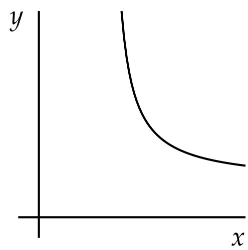

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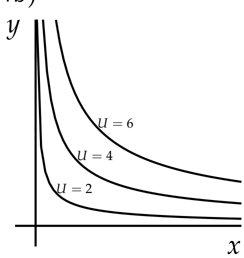
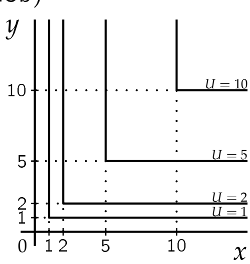
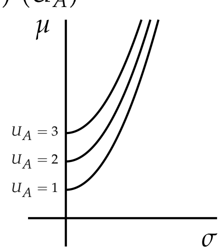


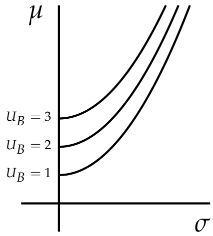
3.5b)



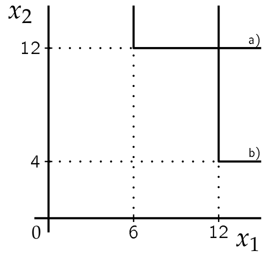
3.6c) (U_A)



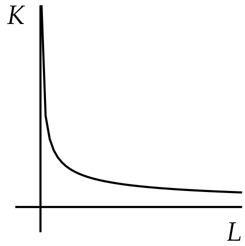
3.6c) (U_B)



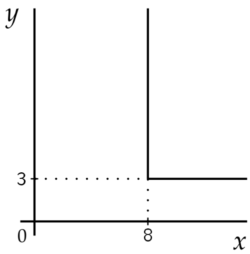
3.10ab)



3.11a)



3.12a)



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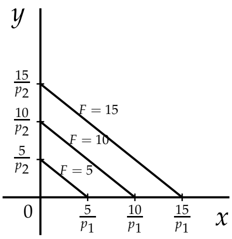

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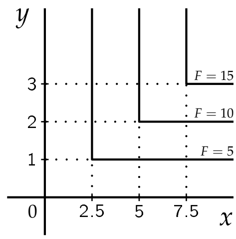

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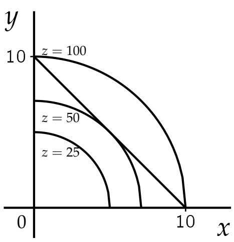

6.12a)