Mathematics for Business Economics

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Preface

This mathematics book is based on many years of experience in mathematics lectures in economics and business administration at Tilburg University, and on teaching high school students. Besides, it is based on the book "*Mathematics with Applications in Micro-Economics*".

As a result, this book perfectly fits the level of first-year students in economics and business administration. Among other things, this is attained by giving more attention to elementary calculation techniques (solving of (in)equalities, arithmetics with powers, etcetera), the change in the order of the mathematical topics and the inclusion of more examples and exercises.

Another important novelty is the **e-learning environment** that has been developed. In this environment, all the mathematical notions and methods are explained by the use of films, text and multiple-choice questions.

New economic applications are also included in the book. Besides applications from microeconomics, such as consumer and producer behavior, attention is also given to modern portfolio theory, inventory management and statistics.

This book would not have been made without the help of a large number of colleagues that supported us with tips and tricks. In particular, we want to mention Ruud Brekelmans, Bart Husslage, Elleke Janssen and Marieke Quant, who all contributed to a great extent to the development of the e-learning environment. Moreover, much additional thanks to Elleke for her support with LaTeX, in particular with respect to the lay-out and the figures.

Herbert Hamers
Bob Kaper
John Kleppe

Tilburg, January 2013

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Introduction

This introduction first presents a short overview of the contents of this book. Next, we discuss the structure of the book. Finally, some recommendations for studying are provided.

In Chapter 1 we introduce functions of one variable. Besides an overview of elementary functions, also the determination of zeros, solving inequalities and calculations using powers are discussed.

Chapter 2 is devoted to the derivative of a function of one variable. Besides a number of differentiation rules, also the inverse function and the economic notions of marginality and elasticity are presented.

In Chapter 3 we introduce functions of two variables and the notion of a level curve. Utility of a consumer and a first introduction to modern portfolio theory are examples of economic applications that will be addressed.

Partial derivatives and partial differentiation are the key elements of Chapter 4. Moreover, the economic notion of marginal rate of substitution is related to the tangent line to a level curve.

Chapter 5 is completely dedicated to the determination of extrema, either with or without constraints. As applications from economics we discuss, among other things, profit maximization of a producer, the Economic Order Quantity model, the regression line, utility maximization of a consumer, cost minimizing of a producer and optimal portfolio selection. Furthermore, also convex and concave functions are discussed, because these functions frequently appear in economic models.

Finally, in Chapter 6 the notion of an integral is introduced and used as tool to determine the area of a region. We apply integrals for the determination of consumer surplus and for the calculation of probabilities.

In this book mathematical and economic topics are discussed in separate sections. In the mathematical sections a mathematical notion or method is followed by a (mathematical) example and an exercise. The example is usually a complete solution or an illustration of the corresponding notion or method; the exercise is a way to fully capture a notion or a method. A similar remark can be made concerning the economic-oriented sections.

Each chapter is concluded with a section of diagnostic exercises. The answers to the exercises are included in the appendix. The complete solutions to the exercises can be found on the website www.academicservice.nl accompanying this book.

The first step in learning mathematics is to visit the lectures that are offered at your university. Understanding mathematics can only be attained by trying to do the exercises yourselves. You will not always manage to solve an exercise. Then it is suggested that you study again the example directly before the exercise. If this provides insufficient information, you have to consult the theory in the book again. Another option is to use the e-learning environment. Here the most important notions and methods are explained in an example by a teacher. One can also practice more with many multiple-choice questions. If you have finished all the exercises in the text, then you can start with the diagnostic exercises of that chapter. These exercises reveal whether you completely understand the chapter.

1

Functions of one variable

Functions are frequently used to describe relations between (economic) variables. In this chapter we introduce the notion of a function of one variable and provide an overview of the classes of polynomial functions, power functions, exponential functions and logarithmic functions. Moreover, elementary algebraic operations such as solving equations and inequalities are discussed.

List of notions

- Function of one variable
- Zero of a function
- Point of intersection of graphs of functions
- Linear function
- Quadratic function
- Polynomial function
- Power function
- Exponential function
- Logarithmic function

1.1 Introduction to functions of one variable

To describe the relation between variables we can use the mathematical notion of a function. In this section we discuss the notion of a function of one variable.

We start this section with an example in which we show how a function can be used to describe the relation between the cost and the quantity of a produced good.

Example 1.1: cost function

A manager has investigated a production process to analyze its costs. The result of this investigation is a relation between the production level q and the corresponding costs C, which can be displayed graphically in a coordinate system (see Figure 1.1). In this coordinate system the horizontal axis represents the production level and the vertical axis represents the corresponding costs. In other words, the independent variable (the production level) is displayed on the horizontal axis and the dependent variable (the costs) is displayed on the vertical axis.

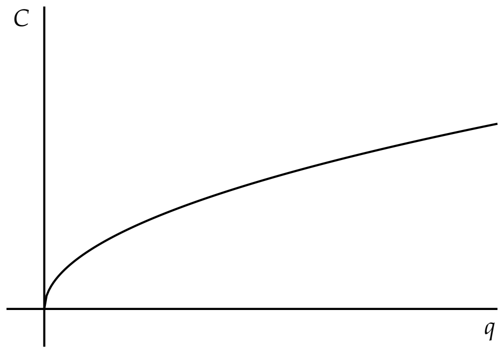

Figure 1.1 The cost function of a production process.

A graphical relation between the production level and the costs is usually not sufficient. A manager wants to know for each production level q the exact corresponding costs $C(q)$. Therefore, it is more useful to describe the relation between the production level and costs by a function: that is a prescription that calculates for each feasible production level q the corresponding costs C. The prescription that fits the graph of Figure 1.1 is $\sqrt{q}$. This prescription means that if the production level q is equal to 9, the costs are $\sqrt{9} = 3$, and if the production level q is equal to 16, the costs are $\sqrt{16} = 4$, etcetera.

Each time the costs are obtained by applying the *prescription* $\sqrt{q}$ to a value of the production level q. Since we assign no interpretation to production levels with a negative value, we restrict the prescription to values q greater than or equal to 0. We conclude that the variables C and q satisfy the equation $C = \sqrt{q}$. We may say that 'the costs C is a function of the production level q'. ◀

If the costs C is a function of the production level q and we do not specify the actual prescription, then we denote the prescription by $C(q)$. The relation between C and q is then given by the equation $C = C(q)$. Hence, in the above example $C(q)$ represents $\sqrt{q}$.

Functions of one variable

A *function of the variable* x is a prescription $y(x)$ which calculates a number, the function value, for any feasible value of the variable x.
The set of all feasible values D of x is called the *domain* of the function.
The set of all possible function values is called the *range* of the function.

If the domain D of a function $y(x)$ is not explicitly given, then the domain consists of all x for which the prescription $y(x)$ can be executed. If additional restrictions are provided, then this will be explicitly stated. Such a restriction, like non-negativity of x, can for example arise from the economic interpretation of the model.

The function values $y(x)$ can be interpreted as the values of a variable. If we call this variable y, then y and x satisfy the equation

$$y = y(x).$$

The variable x in $y(x)$ is called the *independent* or *input variable* and the variable y is called the *dependent* or *output variable*.

Functions and variables are denoted by letters or combinations of letters. Maybe you are used to denoting variables by the letters x, y or z and a function by the letter f (hence $f(x)$ instead of $y(x)$, so that $y = f(x)$). In economic theory, however, one often chooses a letter or a combination of letters that is close to the meaning of the variable (p for **p**rice, L for **L**abour, K (from German '**K**apital') for capital, w for **W**age, etcetera), or of the prescription (MC for **M**arginal **C**ost function, AC for **A**verage **C**ost, etcetera).

The *graph* of a function $y(x)$ is a graphical representation of the function in a coordinate system with two axes, the x-axis and the y-axis, consisting of points with coordinates $(x, y(x))$. It is common to use the horizontal axis for the independent variable x and the vertical axis for the dependent variable y.

Example 1.2: graph of a function

In Figure 1.2 the graph of the function $y(x) = 1 + \sqrt{x+2}$ is shown. The domain of $y(x)$ is $x \geq -2$ and the range is $y \geq 1$. In interval notation the domain is $[-2, \infty)$ and the range $[1, \infty)$.

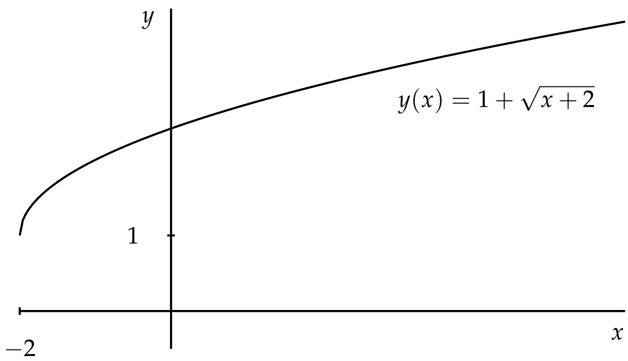

Figure 1.2 *The graph of the function $y(x) = 1 + \sqrt{x+2}$ in an (x, y)-coordinate system.*

A point of intersection of the graph of a function $y(x)$ with the x-axis can be determined by calculating a zero of the function $y(x)$.

Zero of a function of one variable

A *zero* of a function $y(x)$ is a solution of the equation $y(x) = 0$.

A zero a of the function $y(x)$ gives the point of intersection $(a, 0)$ of the corresponding graph and the x-axis.

Point of intersection of two graphs

A point of intersection of the graph $y(x)$ with the graph of another function $z(x)$ is a point (a, b) where a is a solution of the equation $y(x) = z(x)$ and $b = y(a)(= z(a))$.

For the determination of a point of intersection of the graphs of the functions $y(x)$ and $z(x)$ we first calculate the x-coordinate by solving the equation

$$y(x) = z(x).$$

Subsequently, the y-coordinate is obtained by substituting that x-coordinate into one of the two functions.

A point of intersection of the graph of a function $y(x)$ with the y-axis is obtained by calculating the function value at $x = 0$. Hence, a point of intersection with the y-axis is $(0, y(0))$.

Example 1.3: zero and point of intersection

Consider the functions $y(x) = -2x + 2$ and $z(x) = x - 4$. The zero of $y(x)$ is the solution of $y(x) = 0$,

$$\begin{aligned} y(x) = 0 \quad &\Leftrightarrow \quad -2x + 2 = 0 \\ &\Leftrightarrow \quad -2x = -2 \\ &\Leftrightarrow \quad x = 1. \end{aligned}$$

Hence, the point $(1, 0)$ is the point of intersection of the graph of $y(x)$ with the x-axis. A point of intersection of the graph of $y(x)$ with the graph of $z(x)$ is obtained by solving the equation $y(x) = z(x)$,

$$\begin{aligned} y(x) = z(x) \quad &\Leftrightarrow \quad -2x + 2 = x - 4 \\ &\Leftrightarrow \quad -3x = -6 \\ &\Leftrightarrow \quad x = 2. \end{aligned}$$

The x-coordinate of the point of intersection is $x = 2$. Substituting $x = 2$ into the function $y(x)$ we get the y-coordinate, which equals $y(2) = -2$. Obviously, we also have $z(2) = -2$. The point of intersection is $(2, -2)$. Since $y(0) = 2$, the graph of $y(x)$ intersects the y-axis at the point $(0, 2)$. ◀

In the upcoming sections of this chapter we discuss several elementary functions of one variable, show the corresponding graphs, provide some properties and calculate zeros and points of intersection.

1.2 Overview of functions of one variable

1.2.1 Polynomial functions

In this subsection we start with an overview of constant, linear and quadratic functions. Subsequently, we provide the general expression of a polynomial function. In this overview the solving of equations and inequalities is also discussed.

Constant functions

A function of the form

$$y(x) = c,$$

where c is a number, is called a *constant function*. A constant function has the same value for each x. Henceforth, the graph of a constant function is a horizontal line.

Example 1.4: graph of a constant function

An example of a constant function is the function $y(x) = 3$ and its graph is shown in Figure 1.3.

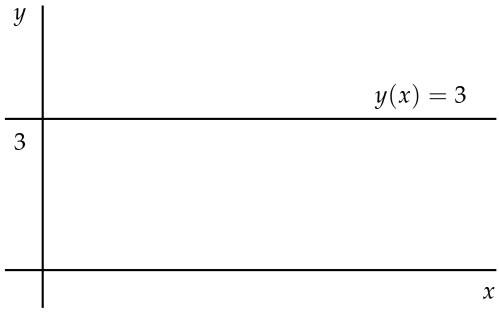

Figure 1.3 The graph of the constant function $y(x) = 3$.

Note that a constant function $y(x) = c$ has no zeros if $c \neq 0$ and that every x is a zero if $c = 0$.

Linear functions

A function of the form

$$y(x) = ax + b,$$

where a and b are numbers ($a \neq 0$), is called a *linear function*. Note that if $a = 0$ then the function $y(x)$ is a constant function. The graph of linear function is a straight line. The *slope* of the line is equal to the number a. The slope of the graph of a linear function indicates the change of the function value if the input x increases by one unit. Since for a linear function this change is always equal to a, it holds that for any x

$$y(x + 1) - y(x) = a.$$

A linear function has a positive slope if $a > 0$ and a negative slope if $a < 0$. Note that the slope of a constant function is equal to 0.

Example 1.5: graph of a linear function

The graphs of the linear functions $y(x) = 3x + 2$ and $z(x) = -2x + 1$ are shown in Figure 1.4 in an $(x, y)-$coordinate system. The slope of the graph of $y(x)$ is equal to 3 and the slope of the graph of $z(x)$ is equal to -2.

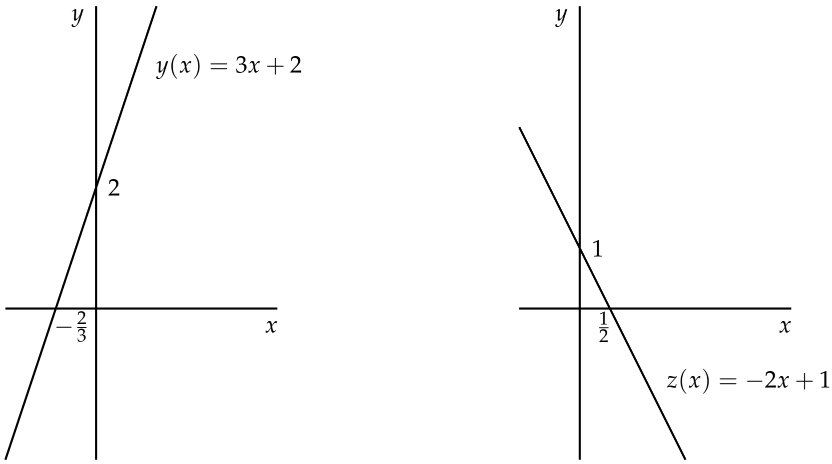

Figure 1.4 *The graph of the function* $y(x) = 3x + 2$ *(left).*
The graph of the function $z(x) = -2x + 1$ *(right).*

In the final example of Section 1.1 we have calculated zeros and points of intersection for linear functions. It is straightforward to determine a general expression for the point of intersection of the graph of a linear function and the x-axis or y-axis, respectively. To determine the point of intersection of the graph of a linear function $y(x) = ax + b$ with the x-axis, we calculate the zero of the function $y(x)$,

$$\begin{aligned} y(x) = 0 \quad &\Leftrightarrow \quad ax + b = 0 \\ &\Leftrightarrow \quad ax = -b \\ &\Leftrightarrow \quad x = -\frac{b}{a}. \end{aligned}$$

Since a linear function $y(x) = ax + b$ has precisely one zero, the graph has precisely one point of intersection with the x-axis. Hence, the point of intersection of the graph of a linear function and the x-axis is $(-\frac{b}{a}, 0)$. The point of intersection with the y-axis is obtained by substituting $x = 0$ into the function: $y(0) = b$. Hence, the point of intersection with the y-axis is $(0, b)$. We can conclude that for a linear function $y(x) = ax + b$ the slope of the line is represented by a, the zero equals $-\frac{b}{a}$ and b is the y-coordinate of the point of intersection of the graph with the y-axis.

Exercise 1.1 (zero and point of intersection)

Consider the functions $y(x) = 3x + 2$ and $z(x) = -5x + 4$.

a) Draw the graphs of the functions $y(x)$ and $z(x)$.
b) Determine the zero of each of the two functions.
c) Determine for the graph of each function the point of intersection with the x-axis.
d) Determine for the graph of each function the point of intersection with the y-axis.
e) Determine the point of intersection of the graphs of these two functions.

Exercise 1.2 (slope of a line)

The points $(2, 4)$ and $(3, 9)$ are on the graph of a linear function $y(x) = ax + b$. Determine a and b.

Quadratic functions

A function of the form

$$y(x) = ax^2 + bx + c,$$

where a, b and c are numbers ($a \neq 0$) is called a *quadratic function*. Note that if $a = 0$ the function is either linear (if $b \neq 0$) or constant (if $b = 0$). The graph of the function $y(x) = ax^2 + bx + c$ is a parabola. This parabola opens upward if $a > 0$ and opens downward if $a < 0$.

Example 1.6: graphs of quadratic functions

The graphs of the quadratic functions $y(x) = -x^2 + 2x + 3$ and $z(x) = 2x^2 + 1$ are shown in Figure 1.5. The graph of the function $y(x)$ is a parabola opened downward because the coefficient of x^2 is equal to -1 (negative), and the graph of the function $z(x)$ is a parabola opened upward because the coefficient of x^2 is equal to 2 (positive).

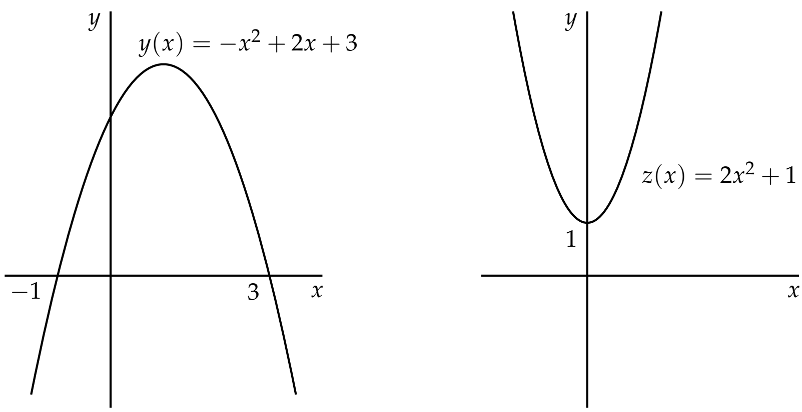

Figure 1.5 *The graph of* $y(x) = -x^2 + 2x + 3$ *is a parabola opened downward (left). The graph of* $z(x) = 2x^2 + 1$ *is a parabola opened upward (right).*

The points of intersection of the graph of a quadratic function $y(x) = ax^2 + bx + c$ and the x-axis are determined by solving the quadratic equation

$$ax^2 + bx + c = 0.$$

A quadratic equation can be solved using the *quadratic formula*. In this formula the *discriminant* is a key component.

The discriminant of a quadratic equation $ax^2 + bx + c = 0$ is equal to $b^2 - 4ac$ and is denoted by D,

$$D = b^2 - 4ac.$$

A quadratic equation has either two, one or zero solutions, depending on the value of the discriminant. The solutions of a quadratic equation are displayed in the following discriminant criterion.

Discriminant criterion and quadratic formula of a quadratic equation

For a quadratic equation $ax^2 + bx + c = 0$ with $a \neq 0$, the following holds for $D = b^2 - 4ac$:

(i) if $D > 0$, then the solutions of the quadratic equation are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(ii) if $D = 0$, then the solution of the quadratic equation is

$$x = -\frac{b}{2a}.$$

(iii) if $D < 0$, then the quadratic equation has no solutions.

Example 1.7: zeros of a quadratic function

Consider the function $y(x) = -x^2 + 2x + 3$. A zero of $y(x)$ is a solution of the equation

$$-x^2 + 2x + 3 = 0.$$

The discriminant is $D = 2^2 - 4 \cdot -1 \cdot 3 = 16$. Because $D > 0$ we obtain according to the quadratic formula the following two solutions:

$$x = \frac{-2 + \sqrt{2^2 - 4 \cdot -1 \cdot 3}}{2 \cdot -1} = -1 \quad \text{and} \quad x = \frac{-2 - \sqrt{2^2 - 4 \cdot -1 \cdot 3}}{2 \cdot -1} = 3.$$

The points of intersection of the graph with the x-axis are $(-1, 0)$ and $(3, 0)$, as can be seen in Figure 1.5.

Exercise 1.3 (zeros of a quadratic function)

Determine the zeros of the following quadratic functions:

a) $y(x) = x^2 + 7x + 6$.

b) $y(x) = 4x^2 + 2x + 1$.

Exercise 1.4 (graph of a quadratic function)

Consider the two functions $y(x) = x^2 + 4x + 3$ and $z(x) = -x^2 + 6$.

a) Draw the graphs of the functions $y(x)$ and $z(x)$.

b) Determine the points of intersection of the graphs of these functions.

When considering two functions the points of intersection are not all that is interesting: also the values of x where the value of one function is greater than or less than the value of the other function could be relevant. To solve this kind of problem we have to solve inequalities. To solve the inequality $f(x) \geq g(x)$ we use the following step plan.

Solving inequality $f(x) \geq g(x)$

Step 1.	Define the function $h(x) = f(x) - g(x)$.
Step 2.	Determine the zeros of $h(x)$.
Step 3.	Make a sign chart of $h(x)$.
Step 4.	Observe in the sign chart where $h(x) \geq 0$.
Conclusion:	The values where $h(x) \geq 0$ are identical to the values where $f(x) \geq g(x)$.

Obviously, we can solve the inequalities $f(x) > g(x), f(x) \leq g(x)$ and $f(x) < g(x)$ in a similar way. In the following example we illustrate the step plan for solving an inequality.

Example 1.8: solving inequalities

Consider the functions $f(x) = x^2 + 2$ and $g(x) = -3x$. We want to find the values of x where $f(x) \geq g(x)$.

Step 1. Define the function $h(x) = f(x) - g(x)$.
We get $h(x) = f(x) - g(x) = x^2 + 2 - (-3x) = x^2 + 3x + 2$.

Step 2. Determine the zero of $h(x)$.

$$\begin{aligned} h(x) = 0 \quad &\Leftrightarrow \quad x^2 + 3x + 2 = 0 \\ &\Leftrightarrow \quad x = -1 \text{ or } x = -2, \end{aligned}$$

where the solutions are obtained by using the quadratic formula.

Step 3. Make a sign chart of $h(x)$.
The sign chart of $h(x)$ is obtained by putting the zeros on a straight line. This divides the line into three intervals: $(-\infty, -2), (-2, -1)$ and $(-1, \infty)$. The function values of $h(x)$ in one interval all have the same sign. This implies that either the function values are all positive or all negative in one interval. Hence, by choosing an arbitrary value in $(-\infty, -2)$ we can determine the sign of $h(x)$ in this interval. For instance, choose $x = -3$. Then $h(-3) = (-3)^2 + 3 \cdot (-3) + 2 = 2 > 0$. Hence, we can conclude that $h(x) > 0$ for all x in the interval $(-\infty, -2)$. Since $h(-1.5) = -0.25$ it holds that $h(x) < 0$ for all x in the interval $(-2, -1)$. Similarly, we find that $h(x) > 0$ for all x in the interval $(-1, \infty)$, because $h(0) = 2$. Figure 1.6 shows the sign chart of $h(x)$.

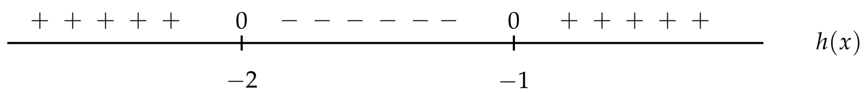

Figure 1.6 Sign chart of $h(x)$.

Step 4. Observe in the sign chart where $h(x) \geq 0$.
From the sign chart it follows that $h(x) \geq 0$ if $x \leq -2$ or $x \geq -1$. Hence $f(x) \geq g(x)$ if $x \leq -2$ or $x \geq -1$.

Exercise 1.5 (solving inequalities)

a) Consider the functions $f(x) = 2x + 4$ and $g(x) = 2x^2 + 3x + 4$. Determine all values of x such that $f(x) \geq g(x)$.
b) Consider the functions $f(x) = x^2 + 4x + 3$ and $g(x) = -x^2 + 6$. Determine all values of x such that $f(x) \geq g(x)$ (see Exercise 1.4).
c) Consider the functions $f(x) = x^2$ and $g(x) = 5x - 4$. Determine all values of x such that $f(x) < g(x)$.

Exercise 1.6 (graph of a quadratic function and the x-axis)

Determine for each of the following functions all values of p such that the graph of the quadratic function has two points of intersection with the x-axis:

a) $y(x) = x^2 + px + 3$.
b) $y(x) = p^2x^2 + 2px + 1$.

Polynomial functions

A function of the form

$$y(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_1 x + a_0$$

where $a_n, a_{n-1}, a_{n-2}, ..., a_1, a_0$ are numbers ($a_n \neq 0$) and n a non-negative integer is called a polynomial function. The degree of a polynomial function is equal to n, which is equal to the highest power in the polynomial function. Constant, linear and quadratic functions are examples of polynomial functions of degree 0, 1 and 2, respectively.

Example 1.9: polynomial function of degree three and zeros

The functions $y(x) = x^3$ and $z(x) = x^3 - 3x^2 + 2x$ are examples of polynomial functions of degree three. The graphs of these functions are shown in Figure 1.7 in an $(x, y)-$coordinate system.

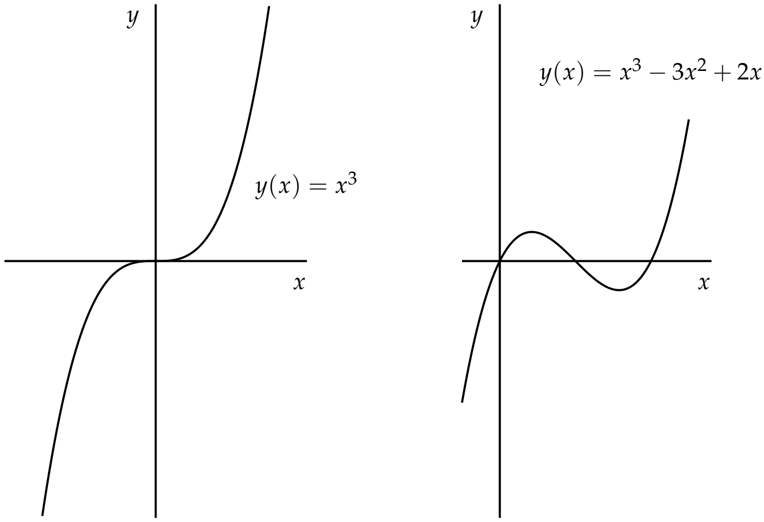

Figure 1.7 *The graph of the function $y(x) = x^3$ (left).*
The graph of the function $z(x) = x^3 - 3x^2 + 2x$ (right).

Obviously, the function $y(x)$ has precisely one zero, which is $x = 0$. The zeros of $z(x)$ can be determined as follows:

$$\begin{aligned} z(x) = 0 \quad &\Leftrightarrow \quad x^3 - 3x^2 + 2x = 0 \\ &\Leftrightarrow \quad x(x^2 - 3x + 2) = 0 \\ &\Leftrightarrow \quad x = 0 \text{ or } x^2 - 3x + 2 = 0 \\ &\Leftrightarrow \quad x = 0, x = 1 \text{ or } x = 2, \end{aligned}$$

where $x^2 - 3x + 2 = 0$ is solved using the quadratic formula. ◀

Exercise 1.7 (zeros of a polynomial function)
Determine the zeros of the following polynomial functions:

a) $y(x) = x^3 - 2x^2 + x$.
b) $y(x) = x^4 - x^2 + x(x^2 - 1)$.
c) $y(x) = 3x^4 - 7x^2 + 2$.

1.2.2 Power functions

A function of the form

$$y(x) = x^k,$$

with k a non-negative integer is called a *positive integer power function*. By definition we have

$$x^0 = 1,$$

and for a positive integer k, x^k means 'multiply x k-times by itself,'

$$x^k = x \cdot x \cdot \ \ldots \ \cdot x \qquad (k\text{-times}).$$

Using positive integer power functions we define a *negative integer power function*:

$$y(x) = x^{-k} = \frac{1}{x^k},$$

with k a positive integer. The graph of a negative integer power function is a hyperbola. Since it is not allowed to divide by zero, the value $x = 0$ is not an element of the domain of such a function.

Example 1.10: graph of a negative integer power function

A frequently used power function is $y(x) = x^{-1}$, or alternatively denoted, $y(x) = \frac{1}{x}$. In Figure 1.8 the graph is drawn of the function $y(x) = \frac{1}{x}$ with $x > 0$.

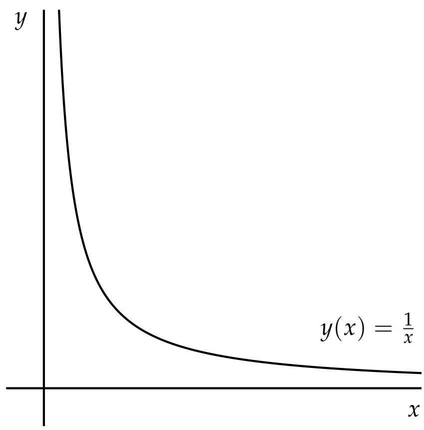

Figure 1.8 The graph of the function $y(x) = x^{-1}$ $(= \frac{1}{x})$, $(x > 0)$.

A function of the form

$$y(x) = x^{\frac{m}{n}}, (x \geq 0),$$

with m and n positive integers and $\frac{m}{n}$ not integral is called a *power function*. An alternative way to represent a power function is

$$y(x) = x^{\frac{m}{n}} = \sqrt[n]{x^m}.$$

A function of the form

$$y(x) = x^{-\frac{m}{n}}, (x > 0),$$

with m and n positive integers and $\frac{m}{n}$ not integer is called a *negative power function.* An alternative way to represent a negative power function is

$$y(x) = x^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{x^m}}.$$

The functions

$$\begin{aligned} y(x) &= x^{\frac{1}{2}} = \sqrt[2]{x^1} = \sqrt{x}, \\ y(x) &= x^{\frac{3}{4}} = \sqrt[4]{x^3}, \\ y(x) &= x^{-\frac{8}{7}} = \frac{1}{\sqrt[7]{x^8}} \end{aligned}$$

are examples of (negative) power functions. In the remaining part of this book the notion power function refers to all types of power functions that are introduced in this section.

Example 1.11: graph of the square root function

A frequently used power function is the square root function. This is the function $y(x) = x^{\frac{1}{2}}$, or alternatively represented, $y(x) = \sqrt{x}$. The graph of the square root function is shown in Figure 1.9.

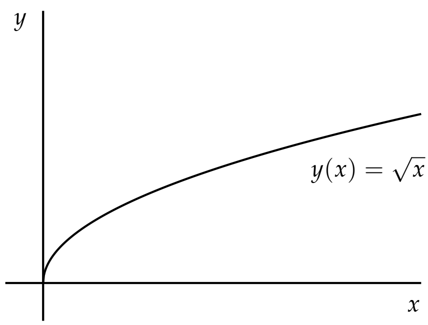

Figure 1.9 The graph of the function $y(x) = x^{\frac{1}{2}} (= \sqrt{x})$.

In the following example we determine the point of intersection of the graph of the square root function and a straight line. In order to do this a new algebraic technique is used.

Example 1.12: point of intersection of the square root function and a straight line

Consider the functions $y(x) = \sqrt{x}$ and $z(x) = x - 2$. We determine all points of intersection of the graphs of these two functions. We solve the following equation:

$$y(x) = z(x) \Leftrightarrow \sqrt{x} = x - 2.$$

This equation seems to be difficult to solve, but by squaring on both sides we obtain

$$x = (x-2)^2 \Leftrightarrow x = x^2 - 4x + 4 \Leftrightarrow x^2 - 5x + 4 = 0.$$

Using the quadratic formula we find that $x = 4$ or $x = 1$. However, we have to be careful, because by squaring on both sides we may have generated solutions that are not a solution of the original equation. Hence, we have to check both solutions in the original equation

$$\sqrt{x} = x - 2.$$

For $x = 4$ we get $2 = 2$, hence we conclude that $x = 4$ is a solution. For $x = 1$ we get $1 = -1$, and we conclude that $x = -1$ is not a solution. Hence, the unique point of intersection of the two graphs is $(x, y) = (4, 2)$. ◀

Power functions satisfy the following properties.

Properties of power functions

1) $x^p \cdot x^q = x^{p+q}$
2) $\dfrac{x^p}{x^q} = x^{p-q}$
3) $(x^p)^q = x^{pq}$
4) $x^p \cdot y^p = (x \cdot y)^p$
5) $x^0 = 1$

Example 1.13: properties of power functions

We determine p such that $(\sqrt[3]{x^5})^{-4} = x^p$. Using the properties of power functions we obtain subsequently

$$(\sqrt[3]{x^5})^{-4} = (x^{\frac{5}{3}})^{-4} = x^{-\frac{20}{3}}.$$

It follows that $p = -\frac{20}{3} = -6\frac{2}{3}$. ◀

Exercise 1.8 (properties of power functions)

Determine p in such a way that the following expressions can be simplified to 2^p:

a) 8.

b) $8^{\frac{4}{3}}$.

c) $\sqrt{32}$.

d) $64^{-\frac{1}{2}}$.

Exercise 1.9 (properties of power functions)

Determine p and q in such a way that the following expressions can be simplified to $x^p y^q$:

a) $x^2 x^5 y y^2$.

b) $\dfrac{x x^{\frac{1}{3}} y^2}{x^{-\frac{2}{3}} y^{-1}}$.

c) $(x^{-1} y^4)^2$.

d) $x^{\frac{10}{6}} \sqrt[3]{x}$.

Exercise 1.10 (properties of power functions)

Solve the following equations:

a) $\frac{x^{\frac{1}{8}}\sqrt[4]{x^3}}{x^{\frac{1}{2}}\sqrt[8]{x^7}} = 2.$ b) $\frac{x^2\sqrt{x}}{8x^{\frac{1}{3}}} = \sqrt[3]{x^2}.$

1.2.3 Exponential functions

A function of the form

$$y(x) = a^x,$$

where a $(a \neq 1)$ is a positive number, is called an *exponential function* with *base* a. The base a can be interpreted as the growth factor per period. In this setting $y(x)$ provides the total factor of multiplication after x periods. Exponential functions where $a > 1$ are used in models in which increasing growth plays an important role.

Example 1.14: graph of an exponential function

The function $y(x) = 2^x$ is an example of an exponential function. Obviously, the growth is increasing since the function values increase rapidly with an increase of x. For instance, at an input of $x = 0$, $x = 1$, $x = 10$ and $x = 20$ the function values are $y(0) = 2^0 = 1$, $y(1) = 2^1 = 2$, $y(10) = 2^{10} = 1024$ and $y(20) = 2^{20} = 1048576$, respectively. The graph of the exponential function $z(x) = (\frac{1}{2})^x$ is obtained by reflecting the graph of $y(x) = 2^x$ into the y-axis. The graphs of both functions are shown in Figure 1.10.

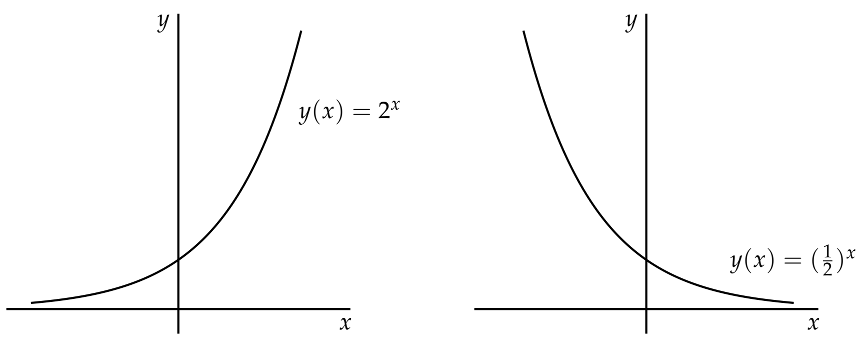

Figure 1.10 The graph of the function $y(x) = 2^x$ (left).
The graph of the function $y(x) = (\frac{1}{2})^x$ (right).

In general the graph of an exponential function $y(x) = a^x$ with base $a > 1$ is similar to the graph of 2^x: the larger the base, the faster the graph approaches the x-axis in the negative x-direction, and the faster the graph increases in the positive x-direction. If the base is $0 < a < 1$, the graph of the function $y(x) = a^x$ is similar to the graph of $(\frac{1}{2})^x$. Note that the graph of any exponential function is above the x-axis. Hence, exponential functions have no zeros.

The exponential function with base e, $y(x) = e^x$, is frequently used in economic growth models. The number e (≈ 2.718) is called the *number of Euler*.

Exponential functions satisfy the following properties.

Properties of exponential functions

1) $a^x \cdot a^y = a^{x+y}$
2) $\dfrac{a^x}{a^y} = a^{x-y}$
3) $(a^x)^y = a^{xy}$
4) $a^x \cdot b^x = (ab)^x$
5) $a^0 = 1$

Exercise 1.11 (properties of exponential functions)

Show by using two of the properties of exponential functions that also the following property holds for exponential functions:

$$a^{-x} = \frac{1}{a^x}.$$

We use the following feature of exponential functions.

Feature of exponential functions

$a^x = a^y$ if and only if $x = y$

Example 1.15: properties and feature of exponential functions

We determine the solution of the equation

$$3^{-x}9^{2x} = 3.$$

Using the properties of exponential functions we can rewrite the left-hand side as follows:

$$3^{-x}9^{2x} = 3^{-x}(3^2)^{2x} = 3^{-x}3^{4x} = 3^{3x}.$$

The right-hand side is equal to 3^1. Hence, the equation $3^{-x}9^{2x} = 3$ can be rewritten as

$$3^{3x} = 3^1.$$

From the feature of exponential functions it follows that

$$3x = 1.$$

Hence $x = \frac{1}{3}$.

Exercise 1.12 (properties and feature of exponential functions)

Solve the following equations:

a) $2^x = 4^{4x+6}$.

b) $27^{2x} = (\frac{1}{3})^{-x+2}$.

c) $(\frac{1}{4})^{x^2-1} = 1$.

Exercise 1.13 (point of intersection of functions)

Consider the functions $y_1(x) = 3^{x+2}$ and $y_2(x) = 24 + 3^x$. Determine the points of intersection of the graphs of these functions.

1.2.4 Logarithmic functions

A function of the form

$$y(x) = {}^a\log x, (x > 0)$$

where a ($a \neq 1$) is a positive number is called a *logarithmic function* with base a. The base a can be interpreted similarly as for an exponential function as a growth factor. In an exponential function $y(x)$ then represents the total multiplication factor after x periods. For a logarithmic function the role of x and $y(x)$ are switched. Here $y(x)$ represents the number of periods that is needed to attain the total multiplication factor x. A logarithmic function is also referred to as logarithm.

Meaning of the logarithmic function with base a

$y(x) = {}^a\log x$ means: to x belongs the y that satisfies $a^y = x$.

Example 1.16: meaning logarithm

For a logarithmic function $y(x) = {}^2\log x$ we have:

$$\begin{aligned} {}^2\log 1 = 0, &\text{ because } 2^0 = 1; \\ {}^2\log 2 = 1, &\text{ because } 2^1 = 2; \\ {}^2\log 8 = 3, &\text{ because } 2^3 = 8. \end{aligned}$$

The definition implies the following relation between the logarithmic function and the exponential function.

Relation between logarithmic and exponential function

A logarithmic and an exponential function with the same base satisfy the following two properties:
$y = {}^{a}\log a^{y}$ and $x = a^{{}^{a}\log x}$.

The above relations imply that the logarithmic function $y(x) = {}^{a}\log x$ is the inverse function of the exponential function of $y(x) = a^{x}$. The inverse function will be discussed extensively in Section 2.5.

Example 1.17: natural logarithm

The logarithm with base e is called the *natural logarithm* and is denoted as

$$y(x) = \ln x.$$

The graph of the natural logarithm is shown in Figure 1.11.

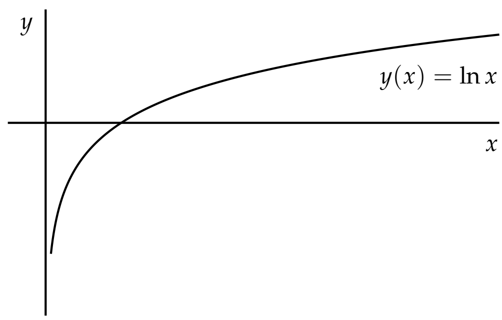

Figure 1.11 The graph of the natural logarithm $y(x) = \ln x$.

Since the natural logarithm is a special case of a logarithmic function, the following two relations are satisfied for the natural logarithm and the exponential function with base e:
(i) $y = \ln e^{y}$,
(ii) $x = e^{\ln x}$, $(x > 0)$.

Moreover, the definition of the logarithm implies that

$$\ln e = 1,$$

since e is the base of the natural logarithm. ◀

We provide four properties of the logarithmic function. For ease of notation we have omitted the base.

Properties of logarithmic functions

1) $\log(x \cdot y) = \log x + \log y$
2) $\log\left(\frac{x}{y}\right) = \log x - \log y$
3) $\log x^y = y \log x$
4) $\log 1 = 0$

Exercise 1.14 (properties of logarithmic functions)

Show by using two of the properties of logarithmic functions that also the following property holds for logarithmic functions:

$$\log\left(\frac{1}{x}\right) = -\log x.$$

As a direct consequence of the feature of exponential functions and the relation between the exponential function and the logarithmic function, we obtain the following feature of logarithmic functions.

Feature of logarithmic functions

${}^a\log x = {}^a\log y$ if and only if $x = y$

Example 1.18: properties and feature of logarithmic functions

We solve the following equation:

$$\ln(x+1) - \ln(x+2) = 1.$$

We rewrite the left-hand side into $\ln(x+1) - \ln(x+2) = \ln(\frac{x+1}{x+2})$. Since $\ln e = 1$ a solution of the original equation is obtained as follows:

$$\begin{aligned} \frac{x+1}{x+2} = e \quad &\Leftrightarrow \quad x+1 = e(x+2) \\ &\Leftrightarrow \quad x+1 = ex + 2e \\ &\Leftrightarrow \quad (1-e)x = 2e-1 \\ &\Leftrightarrow \quad x = \frac{2e-1}{1-e}. \end{aligned}$$

Exercise 1.15 (properties of logarithmic functions)

Rewrite each of the following expressions into one logarithm:

a) $\log x + 2\log y$.   b) $\log x + \log(\frac{1}{y}) - \log z$.

Exercise 1.16 (properties and feature of logarithmic functions)

Solve each of the following equations:

a) $\ln(x+7) + \ln(x+3) = 0$.   b) $({}^3\log x)^2 + 6 = 5\,{}^3\log x$.

In some situations it can be convenient to change the base of a logarithm. Then the following formula can be used.

Changing base of logarithmic functions

$${}^a\log x = \frac{{}^b\log x}{{}^b\log a}$$

Example 1.19: changing base of logarithmic functions

We show that ${}^{100}\log 25 = {}^{10}\log 5$:

$${}^{100}\log 25 = \frac{{}^{10}\log 25}{{}^{10}\log 100} = \frac{2 \cdot {}^{10}\log 5}{2 \cdot {}^{10}\log 10} = {}^{10}\log 5.$$

Exercise 1.17 (changing base of logarithmic functions)

Determine all x such that ${}^{36}\log 81 = {}^6\log x$.

1.3 Applications

In this section we discuss two applications in which elementary functions and their properties are crucial.

1.3.1 Break-even

A company usually wants to maximize its profit. However, in some situations a company is interested in the production level that generates a profit equal to zero, which is the production level where revenue and costs are equal. This production level is called the break-even point. Graphically it is the x-coordinate of the point of intersection of the graph of the revenue function and the graph of the cost function. The following example shows how to determine a break-even point.

Example 1.20: break-even point

Consider the revenue function $R(x) = x$ and the cost function $C(x) = \sqrt{3x+4}$. The break-even point is the point where $R(x)$ equals $C(x)$, in other words, the x-coordinate of the point of intersection (x, y) of the graphs of $R(x)$ and $C(x)$ (see Figure 1.12). Hence, the break-even point is the solution of the equation $R(x) = C(x)$,

$$x = \sqrt{3x+4}.$$

By taking the square on both sides of this equation we obtain

$$x^2 = 3x + 4 \Leftrightarrow x^2 - 3x - 4 = 0.$$

Applying the quadratic formula we get $x = 4$ or $x = -1$. Observe that $x = -1$ is no solution of the equation $R(x) = C(x)$; this solution arises by taking the square on both sides of the original equation. Hence, we can conclude that $x = 4$ is the break-even point at which $R(4) = C(4) = 4$.

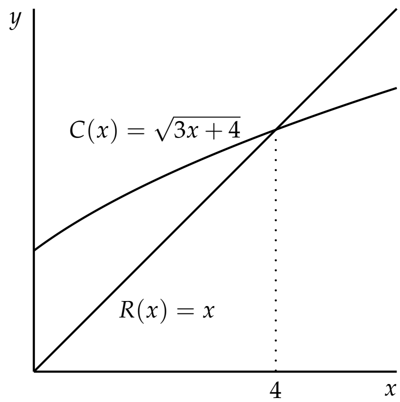

Figure 1.12 The graph of $R(x) = x$ and $C(x) = \sqrt{3x+4}$.

Exercise 1.18 (break-even point)

Consider the revenue function $R(x) = px$, where $p > 0$ and the cost function $C(x) = c + vx$, where $c > 0$, $0 < v < p$. Determine the break-even point.

1.3.2 Market equilibrium

One of the key elements in microeconomics is equilibria. The classical example is the market equilibrium in a demand-supply model. The following example illustrates the determination of such an equilibrium.

Example 1.21: market equilibrium

The demand function of a good is given by $q_d(p) = \frac{1}{p}$, $(p > 0)$ and the supply function is by $q_s(p) = 2p + 1, (p \geq 0)$. The market equilibrium is the point of intersection of the graphs of the demand and supply function. In Figure 1.13 both graphs are shown. Observe that the independent variable p is on the vertical axis. We explain the reason after this example. For the determination of the market equilibrium we have to solve the equation $q_d(p) = q_s(p)$,

$$\frac{1}{p} = 2p + 1.$$

By multiplying both sides by p we obtain

$$1 = 2p^2 + p.$$

We can rewrite this equation into the quadratic equation

$$2p^2 + p - 1 = 0,$$

where the zeros are calculated using the quadratic formula:

$$p = \frac{1}{2} \text{ and } p = -1.$$

The negative value of p is not feasible, hence the equilibrium price is $p = \frac{1}{2}$. The corresponding quantity is $q_s(\frac{1}{2}) = 2(= q_d(\frac{1}{2}))$. Hence, the market equilibrium is $(q, p) = (2, \frac{1}{2})$.

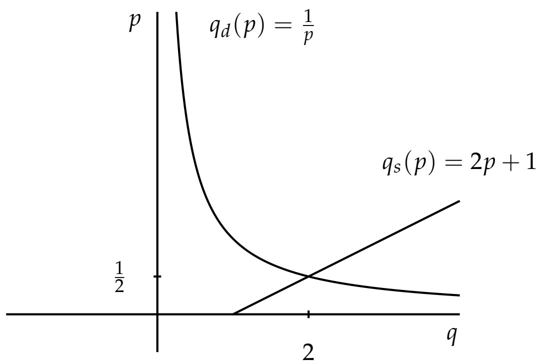

Figure 1.13 The graphs of $q_d(p) = \frac{1}{p}$ and $q_s(p) = 2p + 1$ in a (q, p)-coordinate system.

As stated in the example, for both functions the dependent variable q is displayed on the horizontal axis and the independent variable p on the vertical axis. This is contrary to the mathematical convention with respect to the drawing of a graph. The reason is that at the beginning of the twentieth century the economist Marshall put the price on the vertical axis. This convention is still used in economics, irrespective of whether the price is an independent or dependent variable.

Exercise 1.19 (market equilibrium)

The supply function of good is a linear function. At a price (p) of 220 the demand is 180 units (q). At a price of 160 the demand is 240 units.
a) Determine the demand function.
Also the supply function is linear. At a price of 150 the supply is 100 units and at a price of 250 the supply is 300 units.
b) Determine the supply function.
c) Draw in a (q, p)-coordinate system the graphs of the demand and supply function.
d) Determine the market equilibrium.

1.4 Diagnostic exercises

Exercise 1.20

Determine all a such that the equation $\frac{x}{3} - \frac{a}{x} = 2$ has precisely one solution.

Exercise 1.21

The demand of a good is prescribed by the equation $q = 60 - 10p$. The fixed costs of production of this good are 25 euro and the variable production costs are 2 euro per unit. The total revenue is equal to $TR(p) = pq$. Determine the break-even points.

Exercise 1.22

Solve the inequality $x^3 + 2x \leq 3x^2$.

Exercise 1.23

a) Determine the zeros of the quadratic function $y(x) = 2x^2 + 12x + 18$.
b) Determine p such that the graph of the quadratic function $y(x) = -x^2 - x + p$ has two points of intersection with the x-axis.
c) Consider the functions $y_1(x) = \frac{1}{4}x^2 - 5x + 6$ and $y_2(x) = 3x + p$. Determine all p such that the graphs of the two functions do not intersect.

Exercise 1.24

Consider the functions $y_1(x) = {}^2\log(x-2)$ and $y_2(x) = 2 - {}^2\log(x+4)$.
a) Determine all x such that $y_1(x) > 3$.
b) Determine all x such that $y_1(x) < y_2(x)$.

Exercise 1.25

a) Consider the functions $f(x) = \frac{2-x}{3+x}$ and $g(x) = -x + 1$. Determine all $x > -3$ such that $f(x) \geq g(x)$.
b) Consider the functions $f(x) = \frac{1-x^2}{3+x}$ and $g(x) = x + 1$. Determine all $x > -3$ such that $f(x) \leq g(x)$.

Exercise 1.26

Determine the point of intersection of the graph of the function $y(x) = e^x \ln(x + \frac{1}{2})$ with the x-axis.

Exercise 1.27

Consider the functions $y_1(x) = \sqrt{2x+3}$ and $y_2(x) = x$.

a) Determine the points of intersection of the graphs of these two functions.

b) Solve the inequality $y_1(x) < y_2(x)$.

Exercise 1.28

Solve the following (in)equalities:

a) $8^{2x} \cdot (\frac{1}{4})^{4x^2} = 16 \cdot (\frac{1}{2})^{6x^2}$.

b) $(x^2 - 1)(x + 2) \geq -3(x + 1)$.

c) $2 \cdot {}^3\log(x) < {}^9\log(8x^2) - {}^9\log(81x) + 2$.

Index

Mathematics for Business Economics has been developed for students in mathematics in higher professional and university economics studies. It not only presents the fundamentals of mathematics, but also shows how mathematics is applied, especially in business economics.
It is based on many years of experience in mathematics lectures in economics and business administration at Tilburg University, and on teaching high school students. As a result, this book perfectly fits the level of first-year students in economics and business administration. This is attained by paying more attention to elementary calculation techniques – e.g. solving of (in)equalities, arithmetics with powers –, the change in the order of the mathematical topics and the inclusion of more examples and exercises.

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